Topic Introduction: Primes in Short Intervals; Irregularities of Distribution (the Maier Matrix Method)

Justin Scarfy

The University of British Columbia

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- Hence, we are interested in the functions formed by the difference of two consecutive primes: $p_{n+1} - p_n$, and the number of primes in a given interval $[x, y]$: $\pi(x + y) - \pi(x)$

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Upper Bound: (Baker, Harman, and Pintz 2001)

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$$p_{n+1} - p_n = \mathcal{O}(p_n^{\frac{1}{2} + \frac{1}{40}})$$

Lower Bound: (Pintz 1997)

$$p_{n+1} - p_n > \frac{c \log n \log \log n \log \log \log \log n}{(\log \log \log n)^2}$$

with $c = 2e^{\gamma}$, for infinitely many n

The Irregular Patterns in the Distribution of Primes

Theorem (Maier, 1985)

For any $A > 1$,

$\lim_{n\to\infty}\sup\frac{\pi(n+\log^{A} n)-\pi(n)}{\log^{A-1} n} \geq 1$, $\lim_{n\to\infty}\inf\frac{\pi(n+\log^{A} n)-\pi(n)}{\log^{A-1} n} \leq 1$

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The Proof uses the Maier Matrix:

$$\begin{bmatrix} Qx+1 & Qx+2 & Qx+3 & \dots & Qx+y^C \\ Q(x+1)+1 & Q(x+1)+2 & \dots & \dots & Q(x+1)+y^C \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Q(2x)+1 & Q(2x)+2 & Q(2x)+3 & \dots & Q(2x)+y^C \end{bmatrix}$$

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where y is the variable, $Q = \prod_{p<y} y$, $x = Q^D$,for sufficiently large D, C is to be determined.

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where y is the variable, $Q = \prod_{p<y} y$, $x = Q^D$,for sufficiently large D, C is to be determined. Notice that all the columns form arithmetic progression modulo Q.

Results due to the use of Maier Matrix

The Prime Number Theorem for Arithmetic Progression assures that each column j that is coprime to Q, should contain $\sim \frac{Q}{\phi(Q)} \frac{x}{\log(Qx)}$ primes.

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$$\Phi(y^C, y) \sim y^C \frac{\phi(Q)}{Q} e^{\gamma} \omega(C)$$

where $\omega(C)$ is a function converges to $e^{-\gamma}$, but oscillates above and below $e^{-\gamma}$