

Zeros On the Critical Line I: The Existence of **Infinitely Many** Zeros on the Critical Line

Justin Scarfy

The University of British Columbia



February 21, 2011

Introduction

Although every attempt of proving the Riemann Hypothesis, that all nontrivial zeros of $\zeta(s)$ lie on $\sigma = \frac{1}{2}$, has failed, it is proved by G. H. Hardy in 1914 that $\zeta(s)$ has **infinitely many zeros on $\sigma = \frac{1}{2}$**

Introduction

Although every attempt of proving the Riemann Hypothesis, that all nontrivial zeros of $\zeta(s)$ lie on $\sigma = \frac{1}{2}$, has failed, it is proved by G. H. Hardy in 1914 that $\zeta(s)$ has **infinitely many zeros on $\sigma = \frac{1}{2}$**

Hardy's proof employs the functional equations, $\xi(s)$ and $\Xi(t)$, for the Riemann zeta function $\zeta(s)$: he managed to show the **correspondence of zeros between $\zeta(s)$ on $\sigma = \frac{1}{2}$ and $\Xi(t)$ on the real line**, and proving the existence of infinitely many zeros for $\Xi(t)$ on the real line.

Introduction

Although every attempt of proving the Riemann Hypothesis, that all nontrivial zeros of $\zeta(s)$ lie on $\sigma = \frac{1}{2}$, has failed, it is proved by G. H. Hardy in 1914 that $\zeta(s)$ has **infinitely many zeros on $\sigma = \frac{1}{2}$**

Hardy's proof employs the functional equations, $\xi(s)$ and $\Xi(t)$, for the Riemann zeta function $\zeta(s)$: he managed to show the **correspondence of zeros between $\zeta(s)$ on $\sigma = \frac{1}{2}$ and $\Xi(t)$ on the real line**, and proving the existence of infinitely many zeros for $\Xi(t)$ on the real line.

The Functional Equations for the Zeta Function:

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) \quad (1)$$

$$\Xi(t) := \xi\left(\frac{1}{2} + it\right) = -\frac{1}{2}\left(t^2 + \frac{1}{4}\right)\pi^{-\frac{1}{4} - \frac{1}{2}t}\Gamma\left(\frac{1}{4} + \frac{1}{2}it\right)\zeta\left(\frac{1}{2} + it\right) \quad (2)$$

Lemma 1 (Riemann 1859)

$\xi(s) = \xi(1 - s)$ i.e. the reflexive formula for the functional equation

Lemma 1 (Riemann 1859)

$\xi(s) = \xi(1-s)$ i.e. the reflexive formula for the functional equation

Proof of Lemma 1 (1/3)

By Euler's integral formula, when $\sigma > 0$,

$$\Gamma\left(\frac{1}{2}s\right) = \int_0^\infty e^{-u} u^{\frac{1}{2}s-1} du$$

Lemma 1 (Riemann 1859)

$\xi(s) = \xi(1-s)$ i.e. the reflexive formula for the functional equation

Proof of Lemma 1 (1/3)

By Euler's integral formula, when $\sigma > 0$,

$$\Gamma\left(\frac{1}{2}s\right) = \int_0^\infty e^{-u} u^{\frac{1}{2}s-1} du$$

$$\frac{\Gamma\left(\frac{1}{2}s\right)}{n^s \pi^{\frac{1}{2}s}} = \int_0^\infty e^{-n^2 \pi x} x^{\frac{1}{2}s-1} dx$$

Lemma 1 (Riemann 1859)

$\xi(s) = \xi(1-s)$ i.e. the reflexive formula for the functional equation

Proof of Lemma 1 (1/3)

By Euler's integral formula, when $\sigma > 0$,

$$\Gamma\left(\frac{1}{2}s\right) = \int_0^\infty e^{-u} u^{\frac{1}{2}s-1} du$$
$$\frac{\Gamma\left(\frac{1}{2}s\right)}{n^s \pi^{\frac{1}{2}s}} = \int_0^\infty e^{-n^2 \pi x} x^{\frac{1}{2}s-1} dx$$

Hence if $\sigma > 1$, we sum over n to find:

$$\frac{\Gamma\left(\frac{1}{2}s\right) \zeta(s)}{\pi^{\frac{1}{2}s}} = \sum_{n=1}^{\infty} \int_0^\infty x^{\frac{1}{2}s-1} e^{-n^2 \pi x} dx$$

Lemma 1 (Riemann 1859)

$\xi(s) = \xi(1-s)$ i.e. the reflexive formula for the functional equation

Proof of Lemma 1 (1/3)

By Euler's integral formula, when $\sigma > 0$,

$$\Gamma\left(\frac{1}{2}s\right) = \int_0^\infty e^{-u} u^{\frac{1}{2}s-1} du$$
$$\frac{\Gamma\left(\frac{1}{2}s\right)}{n^s \pi^{\frac{1}{2}s}} = \int_0^\infty e^{-n^2 \pi x} x^{\frac{1}{2}s-1} dx$$

Hence if $\sigma > 1$, we sum over n to find:

$$\frac{\Gamma\left(\frac{1}{2}s\right)\zeta(s)}{\pi^{\frac{1}{2}s}} = \sum_{n=1}^{\infty} \int_0^\infty x^{\frac{1}{2}s-1} e^{-n^2 \pi x} dx = \int_0^\infty x^{\frac{1}{2}s-1} \sum_{n=1}^{\infty} e^{-n^2 \pi x} dx$$

the inversion is justified by absolute convergent

Proof of Lemma 1 (2/3)

$$\psi(x) := \sum_{n=1}^{\infty} e^{-n^2 \pi x} \quad (3)$$

Proof of Lemma 1 (2/3)

$$\psi(x) := \sum_{n=1}^{\infty} e^{-n^2 \pi x} \quad (3)$$

we therefore have, for $\sigma > 1$

$$\zeta(s) = \frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)} \int_0^{\infty} x^{\frac{1}{2}s-1} \psi(x) dx \quad (4)$$

Proof of Lemma 1 (2/3)

$$\psi(x) := \sum_{n=1}^{\infty} e^{-n^2 \pi x} \quad (3)$$

we therefore have, for $\sigma > 1$

$$\zeta(s) = \frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)} \int_0^{\infty} x^{\frac{1}{2}s-1} \psi(x) dx \quad (4)$$

Also we see for $x > 0$,

$$\begin{aligned} \sum_{n=-\infty}^{\infty} e^{-n^2 \pi x} &= \frac{1}{\sqrt{x}} \sum_{n=-\infty}^{\infty} e^{-n^2 \pi \frac{1}{x}} \\ 2 \sum_{n=1}^{\infty} e^{-n^2 \pi x} + e^{-0^2 \pi x} &= \frac{1}{\sqrt{x}} \left(2 \sum_{n=1}^{\infty} e^{-n^2 \pi \frac{1}{x}} + e^{-0^2 \pi \frac{1}{x}} \right) \\ 2\psi(x) + 1 &= \frac{1}{\sqrt{x}} \left(2\psi\left(\frac{1}{x}\right) + 1 \right) \end{aligned} \quad (5)$$

Proof of Lemma 1 (3/3)

The middle equation (4) in the previous slide gives:

$$\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) = \int_0^1 x^{\frac{1}{2}s-1}\psi(x) dx + \int_1^\infty x^{\frac{1}{2}s-1}\psi(x) dx$$

Proof of Lemma 1 (3/3)

The middle equation (4) in the previous slide gives:

$$\begin{aligned}\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) &= \int_0^1 x^{\frac{1}{2}s-1} \psi(x) dx + \int_1^\infty x^{\frac{1}{2}s-1} \psi(x) dx \\ &= \int_0^1 x^{\frac{1}{2}s-1} \left\{ \frac{1}{\sqrt{x}} \psi\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2} \right\} dx + \int_1^\infty x^{\frac{1}{2}s-1} \psi(x) dx\end{aligned}$$

Proof of Lemma 1 (3/3)

The middle equation (4) in the previous slide gives:

$$\begin{aligned}\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) &= \int_0^1 x^{\frac{1}{2}s-1}\psi(x) dx + \int_1^\infty x^{\frac{1}{2}s-1}\psi(x) dx \\ &= \int_0^1 x^{\frac{1}{2}s-1} \left\{ \frac{1}{\sqrt{x}}\psi\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2} \right\} dx + \int_1^\infty x^{\frac{1}{2}s-1}\psi(x) dx \\ &= \frac{1}{s-1} - \frac{1}{s} + \int_0^1 v^{\frac{1}{2}s-\frac{3}{2}}\psi\left(\frac{1}{v}\right) dv + \int_1^\infty x^{\frac{1}{2}s-1}\psi(x) dx\end{aligned}$$

Proof of Lemma 1 (3/3)

The middle equation (4) in the previous slide gives:

$$\begin{aligned}\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) &= \int_0^1 x^{\frac{1}{2}s-1}\psi(x) dx + \int_1^\infty x^{\frac{1}{2}s-1}\psi(x) dx \\&= \int_0^1 x^{\frac{1}{2}s-1} \left\{ \frac{1}{\sqrt{x}} \psi\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2} \right\} dx + \int_1^\infty x^{\frac{1}{2}s-1}\psi(x) dx \\&= \frac{1}{s-1} - \frac{1}{s} + \int_0^1 v^{\frac{1}{2}s-\frac{3}{2}} \psi\left(\frac{1}{v}\right) dv + \int_1^\infty x^{\frac{1}{2}s-1}\psi(x) dx \\&= \frac{1}{s(s-1)} + \int_1^\infty x^{-\frac{1}{2}s-\frac{1}{2}} x^{\frac{1}{2}s-1} \psi(x) dx\end{aligned}$$

Proof of Lemma 1 (3/3)

The middle equation (4) in the previous slide gives:

$$\begin{aligned}\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) &= \int_0^1 x^{\frac{1}{2}s-1}\psi(x) dx + \int_1^\infty x^{\frac{1}{2}s-1}\psi(x) dx \\&= \int_0^1 x^{\frac{1}{2}s-1} \left\{ \frac{1}{\sqrt{x}}\psi\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2} \right\} dx + \int_1^\infty x^{\frac{1}{2}s-1}\psi(x) dx \\&= \frac{1}{s-1} - \frac{1}{s} + \int_0^1 v^{\frac{1}{2}s-\frac{3}{2}}\psi\left(\frac{1}{v}\right) dv + \int_1^\infty x^{\frac{1}{2}s-1}\psi(x) dx \\&= \frac{1}{s(s-1)} + \int_1^\infty x^{-\frac{1}{2}s-\frac{1}{2}}x^{\frac{1}{2}s-1}\psi(x) dx\end{aligned}$$

The last integral is convergent for all values of s .

Proof of Lemma 1 (3/3)

The middle equation (4) in the previous slide gives:

$$\begin{aligned}\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) &= \int_0^1 x^{\frac{1}{2}s-1}\psi(x) dx + \int_1^\infty x^{\frac{1}{2}s-1}\psi(x) dx \\&= \int_0^1 x^{\frac{1}{2}s-1}\left\{\frac{1}{\sqrt{x}}\psi\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2}\right\} dx + \int_1^\infty x^{\frac{1}{2}s-1}\psi(x) dx \\&= \frac{1}{s-1} - \frac{1}{s} + \int_0^1 v^{\frac{1}{2}s-\frac{3}{2}}\psi\left(\frac{1}{v}\right) dv + \int_1^\infty x^{\frac{1}{2}s-1}\psi(x) dx \\&= \frac{1}{s(s-1)} + \int_1^\infty x^{-\frac{1}{2}s-\frac{1}{2}}x^{\frac{1}{2}s-1}\psi(x) dx\end{aligned}$$

The last integral is convergent for all values of s .

By analytic continuation, we see that the R.H.S. is **unchanged** if we replace s by $1-s$, therefore:

$$\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) = \pi^{-\frac{1}{2}(1-s)}\Gamma\left(\frac{1}{2}(1-s)\right)\zeta((1-s))$$

Multiply by $\frac{1}{2}s(s-1)$ yields Lemma 1



Lemma 2

$$\int_0^\infty \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) \cos(xt) dt = \frac{1}{2}\pi \left\{ e^{\frac{1}{2}x} - 2e^{-\frac{1}{2}x} \psi(e^{-2x}) \right\} \quad (6)$$

Lemma 2

$$\int_0^\infty \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) \cos(xt) dt = \frac{1}{2}\pi \left\{ e^{\frac{1}{2}x} - 2e^{-\frac{1}{2}x} \psi(e^{-2x}) \right\} \quad (6)$$

Proof of Lemma 2 (1/2)

This is a special case of which the integral involving $\Xi(t)$ of the form

$$\Phi(x) = \int_0^\infty f(t) \Xi(t) \cos(xt) dt \text{ that can be evaluated.}$$

Where $f(t) := |\phi(it)|^2 = \phi(it)\phi(-it)$, ϕ analytic.

Lemma 2

$$\int_0^\infty \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) \cos(xt) dt = \frac{1}{2}\pi \left\{ e^{\frac{1}{2}x} - 2e^{-\frac{1}{2}x} \psi(e^{-2x}) \right\} \quad (6)$$

Proof of Lemma 2 (1/2)

This is a special case of which the integral involving $\Xi(t)$ of the form

$$\Phi(x) = \int_0^\infty f(t) \Xi(t) \cos(xt) dt \text{ that can be evaluated.}$$

Where $f(t) := |\phi(it)|^2 = \phi(it)\phi(-it)$, ϕ analytic.

Let $y = e^x$:

$$\Phi(x) = \frac{1}{2} \int_{-\infty}^\infty \phi(it)\phi(-it)\Xi(t)y^{it} dt$$

Lemma 2

$$\int_0^\infty \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) \cos(xt) dt = \frac{1}{2}\pi \left\{ e^{\frac{1}{2}x} - 2e^{-\frac{1}{2}x} \psi(e^{-2x}) \right\} \quad (6)$$

Proof of Lemma 2 (1/2)

This is a special case of which the integral involving $\Xi(t)$ of the form

$$\Phi(x) = \int_0^\infty f(t) \Xi(t) \cos(xt) dt \text{ that can be evaluated.}$$

Where $f(t) := |\phi(it)|^2 = \phi(it)\phi(-it)$, ϕ analytic.

Let $y = e^x$:

$$\begin{aligned} \Phi(x) &= \frac{1}{2} \int_{-\infty}^\infty \phi(it)\phi(-it)\Xi(t)y^{it} dt \\ &= \frac{1}{2} \int_{-\infty}^\infty \phi(it)\phi(-it)\xi\left(\frac{1}{2} + it\right)y^{it} dt \end{aligned}$$

Lemma 2

$$\int_0^\infty \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) \cos(xt) dt = \frac{1}{2}\pi \left\{ e^{\frac{1}{2}x} - 2e^{-\frac{1}{2}x} \psi(e^{-2x}) \right\} \quad (6)$$

Proof of Lemma 2 (1/2)

This is a special case of which the integral involving $\Xi(t)$ of the form

$$\Phi(x) = \int_0^\infty f(t) \Xi(t) \cos(xt) dt \text{ that can be evaluated.}$$

Where $f(t) := |\phi(it)|^2 = \phi(it)\phi(-it)$, ϕ analytic.

Let $y = e^x$:

$$\begin{aligned} \Phi(x) &= \frac{1}{2} \int_{-\infty}^\infty \phi(it)\phi(-it)\Xi(t)y^{it} dt \\ &= \frac{1}{2} \int_{-\infty}^\infty \phi(it)\phi(-it)\xi\left(\frac{1}{2} + it\right)y^{it} dt \\ &= \frac{1}{2i\sqrt{y}} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \phi\left(s - \frac{1}{2}\right)\phi\left(\frac{1}{2} - s\right)\xi(s)y^s ds \end{aligned}$$

Proof of Lemma 2 (2/2)

$$\Phi(x) = \frac{1}{2i\sqrt{y}} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \phi\left(s - \frac{1}{2}\right) \phi\left(\frac{1}{2} - s\right) (s-1) \Gamma\left(1 + \frac{1}{2}s\right) \pi^{-\frac{1}{2}s} \zeta(s) y^s ds$$

Proof of Lemma 2 (2/2)

$$\Phi(x) = \frac{1}{2i\sqrt{y}} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \phi\left(s - \frac{1}{2}\right) \phi\left(\frac{1}{2} - s\right) (s - 1) \Gamma\left(1 + \frac{1}{2}s\right) \pi^{-\frac{1}{2}s} \zeta(s) y^s ds$$

Now put $\phi(s) = \left(s + \frac{1}{2}\right)^{-1}$, (so $|\phi(it)|^2 = \left(t^2 + \frac{1}{4}\right)^{-1}$) we have:

$$\Phi(x) = - \frac{1}{2i\sqrt{y}} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{1}{s} \Gamma\left(1 + \frac{1}{2}s\right) \pi^{-\frac{1}{2}s} \zeta(s) y^s ds$$

Proof of Lemma 2 (2/2)

$$\Phi(x) = \frac{1}{2i\sqrt{y}} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \phi\left(s - \frac{1}{2}\right) \phi\left(\frac{1}{2} - s\right) (s-1) \Gamma\left(1 + \frac{1}{2}s\right) \pi^{-\frac{1}{2}s} \zeta(s) y^s ds$$

Now put $\phi(s) = \left(s + \frac{1}{2}\right)^{-1}$, (so $|\phi(it)|^2 = \left(t^2 + \frac{1}{4}\right)^{-1}$) we have:

$$\begin{aligned} \Phi(x) &= -\frac{1}{2i\sqrt{y}} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{1}{s} \Gamma\left(\mathbf{1} + \frac{1}{2}s\right) \pi^{-\frac{1}{2}s} \zeta(s) y^s ds \\ &= -\frac{1}{\mathbf{4}i\sqrt{y}} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{1}{s} \Gamma\left(\frac{1}{2}s\right) \pi^{-\frac{1}{2}s} \zeta(s) y^s ds \end{aligned}$$

Proof of Lemma 2 (2/2)

$$\Phi(x) = \frac{1}{2i\sqrt{y}} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \phi\left(s - \frac{1}{2}\right) \phi\left(\frac{1}{2} - s\right) (s-1) \Gamma\left(1 + \frac{1}{2}s\right) \pi^{-\frac{1}{2}s} \zeta(s) y^s ds$$

Now put $\phi(s) = \left(s + \frac{1}{2}\right)^{-1}$, (so $|\phi(it)|^2 = \left(t^2 + \frac{1}{4}\right)^{-1}$) we have:

$$\begin{aligned} \Phi(x) &= -\frac{1}{2i\sqrt{y}} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{1}{s} \Gamma\left(1 + \frac{1}{2}s\right) \pi^{-\frac{1}{2}s} \zeta(s) y^s ds \\ &= -\frac{1}{4i\sqrt{y}} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{1}{s} \Gamma\left(\frac{1}{2}s\right) \pi^{-\frac{1}{2}s} \zeta(s) y^s ds \\ &= \frac{\pi}{\sqrt{y}} \psi\left(\frac{1}{y^2}\right) + \frac{1}{2} \pi \sqrt{y} \end{aligned}$$

Proof of Lemma 2 (2/2)

$$\Phi(x) = \frac{1}{2i\sqrt{y}} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \phi\left(s - \frac{1}{2}\right) \phi\left(\frac{1}{2} - s\right) (s-1) \Gamma\left(1 + \frac{1}{2}s\right) \pi^{-\frac{1}{2}s} \zeta(s) y^s ds$$

Now put $\phi(s) = \left(s + \frac{1}{2}\right)^{-1}$, (so $|\phi(it)|^2 = \left(t^2 + \frac{1}{4}\right)^{-1}$) we have:

$$\begin{aligned} \Phi(x) &= -\frac{1}{2i\sqrt{y}} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{1}{s} \Gamma\left(1 + \frac{1}{2}s\right) \pi^{-\frac{1}{2}s} \zeta(s) y^s ds \\ &= -\frac{1}{4i\sqrt{y}} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{1}{s} \Gamma\left(\frac{1}{2}s\right) \pi^{-\frac{1}{2}s} \zeta(s) y^s ds \\ &= \frac{\pi}{\sqrt{y}} \psi\left(\frac{1}{y^2}\right) + \frac{1}{2} \pi \sqrt{y} \end{aligned}$$

Inserting back $y = e^x$, we get the desired integral:

$$\int_0^\infty \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) \cos(xt) dt = \frac{1}{2} \pi \left\{ e^{\frac{1}{2}x} - 2e^{-\frac{1}{2}x} \psi(e^{-2x}) \right\}$$



Theorem (Hardy 1914)

The $\zeta(s)$ has infinitely many zeros on $\sigma = \frac{1}{2}$

Theorem (Hardy 1914)

The $\zeta(s)$ has infinitely many zeros on $\sigma = \frac{1}{2}$

Proof of Theorem (1/5)

Since $\Xi(t) = \xi\left(\frac{1}{2} + it\right) = -\frac{1}{2}\left(t^2 + \frac{1}{4}\right)\pi^{-\frac{1}{4} - \frac{1}{2}t}\Gamma\left(\frac{1}{4} + \frac{1}{2}it\right)\zeta\left(\frac{1}{2} + it\right)$ is an **even integrable function** of t (by Lemma 1), and is real for real t .

Theorem (Hardy 1914)

The $\zeta(s)$ has infinitely many zeros on $\sigma = \frac{1}{2}$

Proof of Theorem (1/5)

Since $\Xi(t) = \xi\left(\frac{1}{2} + it\right) = -\frac{1}{2}\left(t^2 + \frac{1}{4}\right)\pi^{-\frac{1}{4} - \frac{1}{2}it}\Gamma\left(\frac{1}{4} + \frac{1}{2}it\right)\zeta\left(\frac{1}{2} + it\right)$ is an even integrable function of t (by Lemma 1), and is real for real t .

A zero of $\zeta(s)$ on $\sigma = \frac{1}{2}$ therefore corresponds to a real zero of $\Xi(t)$, thus it suffices to show that $\Xi(t)$ has infinitely many **real zeros**.

Theorem (Hardy 1914)

The $\zeta(s)$ has infinitely many zeros on $\sigma = \frac{1}{2}$

Proof of Theorem (1/5)

Since $\Xi(t) = \xi\left(\frac{1}{2} + it\right) = -\frac{1}{2}\left(t^2 + \frac{1}{4}\right)\pi^{-\frac{1}{4}-\frac{1}{2}t}\Gamma\left(\frac{1}{4} + \frac{1}{2}it\right)\zeta\left(\frac{1}{2} + it\right)$ is an even integrable function of t (by Lemma 1), and is real for real t .

A zero of $\zeta(s)$ on $\sigma = \frac{1}{2}$ therefore corresponds to a real zero of $\Xi(t)$, thus it suffices to show that $\Xi(t)$ has infinitely many real zeros.

Putting $x = -i\alpha$ in (6), Lemma 2, we have

$$\begin{aligned}\frac{2}{\pi} \int_0^\infty \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) \cosh(\alpha t) dt &= e^{-\frac{1}{2}\alpha} - 2e^{\frac{1}{2}i\alpha}\psi(e^{2i\alpha}) \\ &= 2\cos\frac{1}{2}\alpha - 2e^{\frac{1}{2}i\alpha}\left\{\frac{1}{2} + \psi(e^{2i\alpha})\right\}\end{aligned}$$

Proof of Theorem (2/5)

Since $\zeta(\frac{1}{2} + it) = \mathcal{O}(t^A)$, $\Xi(t) = \mathcal{O}(t^A e^{-\frac{1}{4}\pi t})$, and the last integral may be differentiated w.r.t. α any number of times provided that $\alpha < \frac{1}{4}\pi$.

Proof of Theorem (2/5)

Since $\zeta(\frac{1}{2} + it) = \mathcal{O}(t^A)$, $\Xi(t) = \mathcal{O}(t^A e^{-\frac{1}{4}\pi t})$, and the last integral may be differentiated w.r.t. α any number of times provided that $\alpha < \frac{1}{4}\pi$. Thus,

$$\begin{aligned} \frac{2}{\pi} \int_0^\infty \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) t^{2n} \cosh(\alpha t) dt &= \frac{(-1)^n \cos(\frac{1}{2}\alpha)}{2^{2n-1}} \\ &\quad - 2 \left(\frac{d}{d\alpha}\right)^{2n} e^{\frac{1}{2}i\alpha} \left\{ \frac{1}{2} + \psi(e^{2i\alpha}) \right\} \end{aligned}$$

Proof of Theorem (2/5)

Since $\zeta(\frac{1}{2} + it) = \mathcal{O}(t^A)$, $\Xi(t) = \mathcal{O}(t^A e^{-\frac{1}{4}\pi t})$, and the last integral may be differentiated w.r.t. α any number of times provided that $\alpha < \frac{1}{4}\pi$. Thus,

$$\begin{aligned} \frac{2}{\pi} \int_0^\infty \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) t^{2n} \cosh(\alpha t) dt &= \frac{(-1)^n \cos(\frac{1}{2}\alpha)}{2^{2n-1}} \\ &\quad - 2 \left(\frac{d}{d\alpha}\right)^{2n} e^{\frac{1}{2}i\alpha} \left\{ \frac{1}{2} + \psi(e^{2i\alpha}) \right\} \end{aligned}$$

Next we show that the **last term** tends to 0 as $\alpha \rightarrow \frac{1}{4}\pi$.

Proof of Theorem (2/5)

Since $\zeta(\frac{1}{2} + it) = \mathcal{O}(t^A)$, $\Xi(t) = \mathcal{O}(t^A e^{-\frac{1}{4}\pi t})$, and the last integral may be differentiated w.r.t. α any number of times provided that $\alpha < \frac{1}{4}\pi$. Thus,

$$\begin{aligned} \frac{2}{\pi} \int_0^\infty \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) t^{2n} \cosh(\alpha t) dt &= \frac{(-1)^n \cos(\frac{1}{2}\alpha)}{2^{2n-1}} \\ &\quad - 2 \left(\frac{d}{d\alpha}\right)^{2n} e^{\frac{1}{2}i\alpha} \left\{ \frac{1}{2} + \psi(e^{2i\alpha}) \right\} \end{aligned}$$

Next we show that the last term tends to 0 as $\alpha \rightarrow \frac{1}{4}\pi$.

Again use (5), the property of $\psi(x)$ exploited in Lemma 1:

$$\begin{aligned} \psi(x) &= \frac{1}{\sqrt{x}} \psi\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2} \\ \psi(i + \delta) &= \sum_{n=1}^{\infty} e^{-n^2\pi(i+\delta)} = \sum_{n=1}^{\infty} (-1)^n e^{-n^2\pi\delta} \end{aligned}$$

Proof of Theorem (3/5)

It follows: $\psi(i + \delta) = 2\psi(4\delta) - \psi(\delta) = \frac{1}{\sqrt{\delta}}\psi\left(\frac{1}{4\delta}\right) - \frac{1}{\sqrt{\delta}}\psi\left(\frac{1}{\delta}\right) - \frac{1}{2}$

Hence $\frac{1}{2} + \psi(x)$ and all its derivatives tend to zero as $x \rightarrow i$ along any route in an angle $|\arg(x - i)| < \frac{1}{2}\pi$

Proof of Theorem (3/5)

It follows: $\psi(i + \delta) = 2\psi(4\delta) - \psi(\delta) = \frac{1}{\sqrt{\delta}}\psi\left(\frac{1}{4\delta}\right) - \frac{1}{\sqrt{\delta}}\psi\left(\frac{1}{\delta}\right) - \frac{1}{2}$

Hence $\frac{1}{2} + \psi(x)$ and all its derivatives tend to zero as $x \rightarrow i$ along any route in an angle $|\arg(x - i)| < \frac{1}{2}\pi$

We have thus proved that

$$\lim_{\alpha \rightarrow \frac{1}{4}\pi} \int_0^\infty \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) t^{2n} \cosh(\alpha t) dt = \frac{(-1)^n \pi \cos(\frac{1}{8}\pi)}{2^{2n}} \quad (7)$$

Proof of Theorem (3/5)

It follows: $\psi(i + \delta) = 2\psi(4\delta) - \psi(\delta) = \frac{1}{\sqrt{\delta}}\psi\left(\frac{1}{4\delta}\right) - \frac{1}{\sqrt{\delta}}\psi\left(\frac{1}{\delta}\right) - \frac{1}{2}$

Hence $\frac{1}{2} + \psi(x)$ and all its derivatives tend to zero as $x \rightarrow i$ along any route in an angle $|\arg(x - i)| < \frac{1}{2}\pi$

We have thus proved that

$$\lim_{\alpha \rightarrow \frac{1}{4}\pi} \int_0^\infty \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) t^{2n} \cosh(\alpha t) dt = \frac{(-1)^n \pi \cos\left(\frac{1}{8}\pi\right)}{2^{2n}} \quad (7)$$

Suppose now $\Xi(t)$ were **ultimately of one sign** (for the sake of contradiction), say **positive** (negative can be shown by the same reason) for $t \geq T$, then

$$\lim_{\alpha \rightarrow \frac{1}{4}\pi} \int_T^\infty \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) t^{2n} \cosh(\alpha t) dt = L > 0$$

Proof of Theorem (4/5)

For all $\alpha < \frac{1}{4}\pi$ and $T' > T$,

$$0 < \int_T^{T'} \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) t^{2n} \cosh(\alpha t) dt \leq L$$

Proof of Theorem (4/5)

For all $\alpha < \frac{1}{4}\pi$ and $T' > T$,

$$0 < \int_T^{T'} \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) t^{2n} \cosh(\alpha t) dt \leq L$$

Hence, letting $\alpha \rightarrow \frac{1}{4}\pi$

$$\int_T^{T'} \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) t^{2n} \cosh\left(\frac{1}{4}\pi t\right) dt \leq L$$

Proof of Theorem (4/5)

For all $\alpha < \frac{1}{4}\pi$ and $T' > T$,

$$0 < \int_T^{T'} \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) t^{2n} \cosh(\alpha t) dt \leq L$$

Hence, letting $\alpha \rightarrow \frac{1}{4}\pi$

$$\int_T^{T'} \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) t^{2n} \cosh\left(\frac{1}{4}\pi t\right) dt \leq L$$

Thus the following integral converges due to the L.H.S. of (7), and converges with respect to α for $0 \leq \alpha \leq \frac{1}{4}\pi$:

$$\int_0^\infty \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) t^{2n} \cosh\left(\frac{1}{4}\pi t\right) dt = \frac{(-1)^n \pi \cos\left(\frac{1}{8}\pi\right)}{2^{2n}} \quad (8)$$

for every n .

Proof of Theorem (5/5)

Equation (8) on the previous slide, however, is **impossible**, since when taking n **odd**, the R.H.S. is **negative**, therefore

$$\begin{aligned} \int_T^\infty \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) t^{2n} \cosh\left(\frac{1}{4}\pi t\right) dt &< - \int_0^T \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) t^{2n} \cosh\left(\frac{1}{4}\pi t\right) dt \\ &< KT^{2n} \end{aligned}$$

where K is independent of n .

Proof of Theorem (5/5)

Equation (8) on the previous slide, however, is **impossible**, since when taking n odd, the R.H.S. is **negative**, therefore

$$\begin{aligned} \int_T^\infty \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) t^{2n} \cosh\left(\frac{1}{4}\pi t\right) dt &< - \int_0^T \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) t^{2n} \cosh\left(\frac{1}{4}\pi t\right) dt \\ &< KT^{2n} \end{aligned}$$

where K is independent of n .

But by hypothesis ($\Xi(t) > 0$ for $t \geq T$), there is a **positive** $m = m(T)$ where $\left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) \geq m$ for $2T \leq t \leq 2T+1$:

$$\int_T^\infty \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) t^{2n} \cosh\left(\frac{1}{4}\pi t\right) dt \geq \int_{2T}^{2T+1} m t^{2n} dt \geq m(2T)^{2n}$$

Proof of Theorem (5/5)

Equation (8) on the previous slide, however, is **impossible**, since when taking n odd, the R.H.S. is negative, therefore

$$\int_T^\infty \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) t^{2n} \cosh\left(\frac{1}{4}\pi t\right) dt < - \int_0^T \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) t^{2n} \cosh\left(\frac{1}{4}\pi t\right) dt \\ < \mathbf{KT^{2n}}$$

where K is independent of n .

But by hypothesis ($\Xi(t) > 0$ for $t \geq T$), there is a positive $m = m(T)$ where $\left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) \geq m$ for $2T \leq t \leq 2T + 1$:

$$\int_T^\infty \left(t^2 + \frac{1}{4}\right)^{-1} \Xi(t) t^{2n} \cosh\left(\frac{1}{4}\pi t\right) dt \geq \int_{2T}^{2T+1} m t^{2n} dt \geq \mathbf{m(2T)^{2n}}$$

Hence, $\mathbf{m2^{2n}} < K$, which is false for sufficiently large n . □