Zeros On the Critical Line II: A Finite Proportion of the Zeros lie on the Critical Line

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February 23, 2011

Introduction

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Selberg's (1942b) proof modifies Hardy and Littlewood's (1921) proof by employing a general case of the Fourier transformations used in their (1921) paper. He then maneuvered these transforms to prove this Theorem

Fourier Transformations

Let $F(u), f(y)$ be functions related by the Fourier formulae

$$F(u) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(y) e^{iyu}\, dy \qquad f(y) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} F(u) e^{-iyu}\, du$$

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Integrating over $(t, t+H)$, we obtain

$$\int_t^{t+H} F(u)\, du = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(y) \frac{e^{iyH} - 1}{iy} e^{iyt}\, dy,$$

so that $\int_t^{t+H} F(u)\, du$ and $f(y)\frac{e^{iyH} - 1}{iy}$ are Fourier transforms.

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Parseval's Theorem of the normal form gives (for $F(u)$ real, $f(y)$ even):

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$$\leq 2H^2 \int_0^{1/H} |f(y)|^2\, dy + 8 \int_{1/H}^{\infty} \frac{|f(y)|^2}{y^2}\, dy \qquad (1)$$

Some Definitions Selberg used to Improve the Result

Define α_ν by

$$\frac{1}{\sqrt{\zeta(s)}} = \sum_{\nu=1}^{\infty} \frac{\alpha_\nu}{\nu^s} \quad (\sigma > 1), \quad \alpha_1 = 1$$

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$$\beta_\nu := \alpha_\nu \left(1 - \frac{\log \nu}{\log X}\right) \qquad (1 \le \nu < X)$$

Hence, all sums involving β_ν run over $[1, \, X]$ become:
(as we may assume $\beta_\nu = 0$ for $\nu \ge X$)

$$\phi(s) := \sum \frac{\beta_\nu}{\nu^s}$$

Preliminaries 1/3

$$\Phi(z) := \frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma\left(\frac{1}{2}s\right)\pi^{-\frac{1}{2}s}\zeta(s)\phi(s)\phi(1-s)z^s\, ds \qquad \text{where } c > 1$$

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On the other hand, by definition of ϕ:

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Preliminaries 2/3

Putting $z = e^{-i(\frac{1}{4}\pi - \frac{1}{2}\delta) - y}$, it follows that the functions:

$$F(t) = \frac{1}{\sqrt{(2\pi)}} \frac{\Xi(t)}{t^2 + \frac{1}{4}} \left| \phi\left(\frac{1}{2} + it\right) \right|^2 e^{(\frac{1}{4}\pi - \frac{1}{2}\delta)t} \tag{3}$$

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Inserting $y = \log x$, $G = e^{1/H}$ in (1) with $F(t)$ and $f(y)$ defined above for $H \leq 1$, the first integral on the right equals to

$$\int_1^G \left| \frac{e^{-i(\frac{1}{4}\pi - \frac{1}{2}\delta)}}{2x} \phi(1)\phi(0) - \sum_{n=1}^{\infty} \sum_{\mu} \sum_{\nu} \frac{\beta_\mu \beta_\nu}{\nu} \exp\left(-\frac{\pi n^2 \mu^2}{z^2 \nu^2} e^{i(\frac{1}{2}\pi - \delta)} x^2\right) \right|^2 dx$$

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Calling the triple sum $g(x)$, then the above equation is not greater than:

$$2 \int_1^G \frac{|\phi(1)\phi(0)|^2}{4x^2} dx + 2 \int_1^G |g(x)|^2 dx < \frac{1}{2} |\phi(1)\phi(0)|^2 + 2 \int_1^G |g(x)|^2 dx$$

Preliminaries 3/3

$$g(x) := \sum_{n=1}^{\infty} \sum_{\mu} \sum_{\nu} \frac{\beta_\mu \beta_\nu}{\nu} \exp\left(-\frac{\pi n^2 \mu^2}{z^2 \nu^2} e^{i(\frac{1}{2}\pi - \delta)} x^2 \right)$$

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Similarly, we can show that the second integral in (1) does not exceed

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We thus obtained upper bounds for integrals (1) as $\delta \to 0$, but if we consider:

$$J(x,\, \theta) := \int_x^\infty |g(u)|^2 u^{-\theta}\, du \qquad (0 < \theta \leq \frac{1}{2},\ 1 \leq x)$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{\kappa\lambda\mu\nu} \frac{\beta_\kappa \beta_\lambda \beta_\mu \beta_\nu}{\lambda\nu} \int_x^\infty \exp\Big\{ -\pi\Big(\frac{m^2\kappa^2}{\lambda^2} + \frac{n^2\mu^2}{\nu^2}\Big) u^2 \sin\delta$$

$$+ i\pi\Big(\frac{m^2\kappa^2}{\lambda^2} - \frac{n^2\mu^2}{\nu^2}\Big) u^2 \cos\delta \Big\} \frac{du}{u^\theta}$$

and let $\sum_1$ denote the sum of those terms in which $m\kappa/\lambda = n\mu/\nu$, and $\sum_2$ the remainder.

Estimates of $\sum_1$ and $\sum_2$ for $J(x, \theta)$ converges uniformly

$$\sum_1 := \sum_{m=1}^{\infty} \sum_{\kappa\lambda\mu\nu} \frac{\beta_\kappa \beta_\lambda \beta_\mu \beta_\nu}{\lambda\nu} \int_x^\infty \exp\left\{ -2\pi \left(\frac{m^2 \kappa^2}{\lambda^2} \right) u^2 \sin\delta \right\} \frac{du}{u^\theta}$$

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It is shown that

$$\sum\nolimits_1 = \mathcal{O}\left(\frac{1}{\delta^{\frac{1}{2}\theta} x^\theta \log X}\right) + \mathcal{O}\left\{\frac{(\delta^{\frac{1}{2}} x X^2)^\theta}{\delta^{\frac{1}{2}} \theta x^\theta \log X}\right\} + \mathcal{O}\left\{\frac{x^{1-\theta}\log(X/\theta)}{\theta} X^2 \log^2 X\right\}$$

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and for $X = \delta^{-c}$, with $0 \le c \le \frac{1}{8}$

$$\sum\nolimits_2 = \mathcal{O}\left\{\frac{1}{x^\theta} \sum_{\kappa\lambda\mu\nu} \left(\frac{\lambda}{\kappa} \log\frac{1}{\delta} + \frac{1}{\kappa\mu} \log^2\frac{1}{\delta}\right)\right\} = \mathcal{O}\left(\frac{X^4}{x^\theta} \log^2\frac{1}{\delta}\right)$$

Estimates for integrals over the Fourier formulae

Lemma 3

Under the assumption of the estimate of $\sum_1$ and $\sum_2$ from the previous slide, we can deduce that

$$\int_{-\infty}^{\infty} \left| \int_{t}^{t+H} F(u)\, du \right|^2 dt = \mathcal{O}\left(\frac{h}{\delta^{\frac{1}{2}} \log X} \right) \tag{5}$$

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$$\int_{-\infty}^{\infty} \left| \int_t^{t+H} F(u)\, du \right|^2 dt = \mathcal{O}\Big(\frac{h}{\delta^{\frac{1}{2}} \log X}\Big) \tag{5}$$

Lemma 4

Similarly, one can show that:

$$\int_{-\infty}^{\infty} \left\{ \int_t^{t+H} |F(u)|\, du \right\}^2 dt = \mathcal{O}\Big(\frac{h^2 \log(1/\delta)}{\delta^{\frac{1}{2}} \log X}\Big) \tag{6}$$

Estimates for integrals over the Fourier formulae

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The proofs of these Lemmas use the property that $J(x,\, \theta)$ is uniformly convergent with respect to θ, and then taking special values of the x's and the θ's.

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Proof of Theorem (2/3)

By Lemma 4 with $\delta = 1/T$, we obtain an upper bound:

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By Lemma 3, we see that

$$\int_0^T \left\{ \int_t^{t+h} |F(u)|\,du - \left| \int_t^{t+h} F(u)\,du \right| \right\} dt \le \left\{ \int_0^T dt \int_0^T \left| \int_t^{t+h} F(u)\,du \right|^2 dt \right\}^{\frac{1}{2}}$$
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Proof of Theorem (3/3)

Therefore, by (7), (8), (9)

$$\{m(E)\}^{\frac{1}{2}} > A_1 T^{\frac{1}{2}} \left(\frac{\log X}{\log T}\right)^{\frac{1}{2}} - A_2 \frac{T^{\frac{1}{2}}}{h^{\frac{1}{2}} \log^{\frac{1}{2}} T},$$

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Hence, of the partitions $(0, h)$, $(h, 2h)$, ... contained in $(0, T)$, at least $[A_3 T/h]$ must contain points of E. If $(nh, (n+1)h)$ contains a point t of E, a zero of $\zeta(\frac{1}{2} + iu)$ must be in $(t, t+h)$, and so does $(nh, (n+2)h)$

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Recall the fact that each zero might be counted twice this way, there must be at least $\frac{1}{2}[A_3 T/h] > AT \log T$ zeros in $(0, T)$ □