Infinitely Many Non-vanishing Dirichlet L-functions at the Critical Point

Justin Scarfy

The University of British Columbia

March 28, 2011

Background

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Jutila's proof involves using the (approximate) functional equation for $L(s, \chi)$, and comparing the mean and the mean square to the averages of

$$\sum_{\chi \bmod q} L(\frac{1}{2}, \chi)$$

where χ is defined modulo a prime q, and then establish an asymptotic formula for the sum $L(s, \chi_D)$, where χ_D is the real characters given by $\left(\frac{D}{\cdot}\right)$

Prelimaries

An Approximate Functional Equation for $L(\frac{1}{2}, \chi)$

When $\Re(s) > 1$, it is obvious there is some integer N with

$$L(s, \chi) = \sum_{n<Nq} \chi(n) n^{-s} + \sum_{n>Nq} \chi(n) n^{-s}$$

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Revisiting the Pólya-Vinogradov Inequality

Recall the Pólya-Vinogradov inequality states that:

$$\sum_{y<n\leq x} \chi(n) \ll q^{\frac{1}{2}} \log q$$

Improving the Estimates

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Their estimates are obtained in the following two lemmas:

Lemma 1

$$\sum_{0<d\leq Y}^{*} L(\frac{1}{2}, \chi_d) = c_1 Y \log Y + c_2 Y + O(Y^{\frac{3}{4}+\epsilon})$$

where the sum is over fundamental discriminates.

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Lemma 2

$$\sum_{0<d\leq Y}^{*} L(\frac{1}{2}, \chi_d)^2 = c_3 Y (\log Y)^3 + O(Y (\log Y)^{\frac{5}{2}+\epsilon})$$

Average value of $L(\frac{1}{2}, \chi_D)$

Theorem

Let $N(Y)$ denote the number of fundamental discriminates $0 < d \leq Y$ such that $L(\frac{1}{2}, \chi) \neq 0$, then

$$N(Y) \gg Y/\log Y$$

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Proof of the Theorem

By Cauchy-Schwarz,

$$\Big| \sum_{0<d\leq Y}^{*} L(\frac{1}{2}, \chi_d) \Big|^2 \ll \Big(\sum_{0<d\leq Y}^{*} L(\frac{1}{2}, \chi_d)^2 \Big) \Big(\sum_{0<d\leq Y}^{*} 1 \Big)$$

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Lemmas 1 and 2 imply that

$$Y^2(\log Y)^2 \ll Y(\log Y)^3 N(Y)$$

□

Set up and the Functional Equation for $L(\frac{1}{2}, \chi)$

Proof of Lemma 1 (1/6)

Let

$$f_Y(n, w) := \sum_{0<d\le Y}^{*} \left(\frac{d}{n}\right) d^w$$

$$f_Y(n) := f_Y(n, 0) = \sum_{0<d\le Y}^{*} \left(\frac{d}{n}\right)$$

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For a real primitive character $\chi \bmod q > 1$ and an $X \geq 1$ we have the identity follows easily from the functional equation:

$$L(\frac{1}{2},\chi) = \sum_{n=1}^{\infty} \chi(n)\exp(-n/X)n^{-\frac{1}{2}}$$
$$- \frac{1}{2\pi i}\int_{(-\frac{1}{2}-\epsilon)} L(\frac{1}{2}-s,\chi)\left(\frac{q}{\pi}\right)^{-s} \frac{\Gamma(\frac{1}{2}(a+\frac{1}{2}-s))}{\Gamma(\frac{1}{2}(a+\frac{1}{2}+s))\Gamma(s)X^s}\, ds$$

where $a = \frac{1}{2}(1-\chi(-1))$

Definitions of S and I

Proof of Lemma 1 (2/5)

If we sum over χ corresponding to $d > 0$, and observing that $a = 0$ in this case, we obtain:

$$\sum_{0<d\leq Y}^{*} L(\frac{1}{2}, \chi_d) = \sum_{n=1}^{\infty} f_Y(n) \exp(-n/X) n^{-\frac{1}{2}}$$

$$- \frac{1}{2\pi i} \int_{(-\frac{1}{2}-\epsilon)} \sum_{n=1}^{\infty} \left(f_Y(n, -s) n^{s-\frac{1}{2}} \right) \pi^s \frac{\Gamma(\frac{1}{4} - \frac{s}{2})}{\Gamma(\frac{1}{4} + \frac{s}{2})} \Gamma(s) X^s \, ds$$

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When n is a square, we have

$$f_Y(n) = c_n Y + O(Y^{\frac{1}{2}} d(n))$$

with

$$c_n = \frac{3}{\pi^2} \prod_{p|n} (1 + \frac{1}{p})^{-1}$$

Estimates for S

Proof of Lemma 1 (3/5)

For $\Re(s) > 0$,

$$f_Y(n,s) = c_n \frac{Y^{1+s}}{1+s} + O((|s|+1)d(n)Y^{\frac{1}{2}+\sigma})$$

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Apply this to S: the square integers n contribute an amount:

$$\sum_{m=1}^{\infty}(c_m Y + O(Y^{\frac{1}{2}}d(m^2)))\exp\left(-m^2/X\right)m^{-1}$$

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And the non-square integers contribute

$$\sum_{1\le n}{}' f_Y(n)\exp(-n/X)n^{-\frac{1}{2}} \ll Y^{\frac{1}{2}}X^{\frac{1}{2}}(\log X)^{\frac{5}{2}}$$

Estimates for I

Proof of Lemma 1 (4/5)

In I, the squares contribute:

$$-\frac{1}{2}\int_{(-\frac{1}{2}-\epsilon)}\frac{Y^{1-s}}{1-s}\frac{3}{\pi^2}\prod_p\left(1-\frac{1}{(p+1)p^{1-2s}}\right)$$

$$\zeta(1-2s)\pi^s\frac{\Gamma(\frac{1}{4}-\frac{s}{2})}{\Gamma(\frac{1}{4}+\frac{s}{2})}\Gamma(s)X^s\,ds+O(Y^{1+\epsilon}X^{-\frac{1}{2}-\epsilon})$$

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Calculation reveals that $\zeta(1-2s)\Gamma(s)=-\frac{1}{2s^2}+\frac{3\gamma}{2s}-\gamma^2+\dots$

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So the main term above is

$$-Y\{c''+\frac{3}{\pi^2}\frac{1}{2}\prod_p\left(1-\frac{1}{(p+1)p}\right)\log X/Y + O((X/Y)^{\frac{1}{2}-\epsilon}\epsilon^{-1})\}$$

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One can show the non-squares' contribution is

$$\ll \epsilon^{-1}Y^{1+\epsilon}X^{-\frac{1}{2}-\epsilon}$$

Proof of Lemma 1 (5/5)

Adding up all the squares and the non-squares portions, we have:

$$S = \frac{3}{2\pi^2} \prod_p \left(1 - \frac{1}{p(p+1)}\right) Y \log X + c' Y + O(Y^{\frac{1}{2}} X^{\frac{1}{2}} (\log X)^{\frac{5}{2}}$$

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An Upper Bound for $\sum^*_{0<d\leq Y} L(\frac{1}{2}, \chi_d)$

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$$+ O(\epsilon^{-1} Y^{1+\epsilon} X^{-\frac{1}{2}-\epsilon})$$

Choosing $X = Y^{\frac{1}{2}}$ gives Lemma 1 □

A Lower Bound for $\sum^*_{0<d\leq Y} L(\frac{1}{2}, \chi_d)^2$

Outline of Proof to Lemma 2 (1/2)

Again from the functional equation:

$$\begin{aligned}
&\sum^*_{0\leq d<Y} L(\frac{1}{2}, \chi_d)^2 \\
&= \sum_{n=1}^{\infty} f_Y(n) \exp(-n/X) n^{-\frac{1}{2}} \\
&\quad - \frac{1}{2\pi i} \int_{(-3/4)} \left\{ \sum_{n=1}^{\infty} f_Y(n, -2s) d(n) n^{s-\frac{1}{2}} \right\} \frac{\Gamma^2(\frac{1}{4} - \frac{s}{2})}{\Gamma^2(\frac{1}{4} + \frac{s}{2})} \Gamma(s) (\pi^2 X)^s \, ds \\
&:= S(X, Y) + I(X, Y)
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\end{aligned}$$

Use the same idea as in the proof of Lemma 1, carefully estimate both the square and non-squares contributions to $S(X,Y)$ and $I(X,Y)$, and then add them up.

Estimates for $S(X, Y)$ and $I(X, Y)$

Outline of Proof to Lemma 2 (2/2)

By using the analogous method described in Lemmas 1's proof, except with more estimates with double sums where the outer ranges over square (and non-square) integers n, and the inner over fundamental discriminants — we obtain estimates:

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for any $A > 0$

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$$c = \frac{1}{8\pi^2} \prod_p \left(1 - \frac{4p^2 - 3p + 1}{p^4 + p^3}\right) \neq 0$$

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Here choose $X = Y$ yields the result □