Hecke L-function, Artin L-function, and their connections with other L-fuctions

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Outline

Introduction

During the seminars in this week, we developed the basic tools such as adèles and idèles, reviewed the classical and disputed over the modern theory of Fourier Analysis, picked up on a few theorems on classical field theory, and learned the functional equations for the Hecke L-functions

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In this concluding lecture of our three days seminar, we shall lightly investigate the interplays between different L-function, namely those of Riemann, Dirichlet, Hurwitz, Dedekind, Hecke, and Artin. Moreover, we shall get a glimpse of the power of Langlands Program which was meant to unite and (extent) them all.

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Disclaimer

The following investigation is by NO means complete, those left out include mostly the ones named after things (e.g. L-Functions of Elliptic Curves)

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$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

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- $\zeta(s)$ has a completed L-function (Riemann version):

$$\Lambda(s) := \pi^{-\frac{s}{2}} s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

Riemann L-function (2/2)

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Riemann Hypothesis

Every non-trivial zero of $\zeta(s)$, when written $s = \sigma + it$, lines on the vertical line $\sigma = \frac{1}{2}$

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Let $s \in \mathbb{C}$, χ be a primitive Dirichlet character modulo q.

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

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- $L(s,\chi)$ has a completed L-function:
  Let $\kappa := \frac{1}{2}(1-\chi(-1))$ (κ equals 0 if χ is even, and 1 when χ is odd)

$$\Lambda(s,\chi) := \left(\frac{\pi}{q}\right)^{-\frac{1}{2}(s+\kappa)} \Gamma\Big(\frac{1}{2}(s+\kappa)\Big) L(s,\chi)$$

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Known properties of this L-function

- $L(s,\chi)$ also has a functional equation:

$$\Lambda(1-s,\bar{\chi}) = \frac{i^{\kappa}\sqrt{q}}{\tau(\chi)} \Lambda(s,\chi)$$

where $\tau(\chi)$ is the Gauss sum attached to χ:

$$\tau(\chi) := \sum_{x \pmod{q}} \chi(x) \exp(2i\pi x/q)$$

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Generalized Riemann Hypothesis

All zeros of $L(s,\chi)$ lie on the line $\sigma = \frac{1}{2} \quad (s = \sigma + it)$

Dedekind L-function (1/3)

Definition

Let K be a number field with discriminant Δ_K,so that $K/\mathbb{Q}$ is a finite field extension of degree $d = [K : \mathbb{Q}]$. Denote $\mathcal{O}_K$ the ring of integers in K. For every ideal $\mathfrak{a} = \mathfrak{p}_1^{a_1} \dots \mathfrak{p}_t^{a_t}$, where the norm is given by $N_\mathfrak{a} = [\mathcal{O}_K : \mathfrak{a}]$, we define the Dedekind L-function by

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$$\zeta_K(s) = \prod_{\mathfrak{p}} \left(1 - \frac{1}{(N\mathfrak{p})^s}\right)^{-1}$$

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- Remark: if $K = \mathbb{Q}$, we recover the Riemann L-function, i.e. $\zeta_\mathbb{Q}(s) = \zeta(s)$

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- Recall the field K has $d = r_{\mathbb{R}} + 2r_{\mathbb{C}}$ embeddings $\sigma : K \hookrightarrow \mathbb{C}$, $r_{\mathbb{R}}$ real embeddings and $r_{\mathbb{C}}$ pairs of complex embeddings. Hecke used this ingredient and proved an analytic continuation of $\zeta_K(s)$ to the complex plane:

$$\Gamma_{\mathbb{R}}(s) := \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)$$
$$\Gamma_{\mathbb{C}}(s) := 2(2\pi)^s\Gamma(s)$$

  And the completed L-function here takes the form (Hecke version)

$$\Lambda_K(s) := |\Delta_K|^{\frac{s}{2}}\Gamma_{\mathbb{R}}(s)^{r_{\mathbb{R}}}\Gamma_{\mathbb{C}}(s)^{r_{\mathbb{C}}}\zeta_K(s)$$

- Hence $\zeta_K(s)$ admits to a (surprisingly simple) functional equation:

$$\Lambda_K(s) = \Lambda_K(1-s)$$

Dedekind L-function (3/3)

Known properties for Dedekind L-function

- Hecke also showed that $\zeta_K(s)$ has only a simple pole at $s = 1$, and

$$\lim_{s \to 1} \zeta_K(s) = \frac{2^{r_{\mathbb{R}}}(2\pi)^{r_{\mathbb{C}}} hR}{\omega \sqrt{|\Delta_K|}}$$

  here h denotes the class number of K;

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  By the Dirichlet's unit theorem, $\mathcal{O}_K^{\times}$ is finitely generated abelian group, thus it has a finite part and a free part, $W \oplus \mathbb{Z}^r$ with $r = r_{\mathbb{R}} + r_{\mathbb{C}} - 1$, the finite part with rank $\omega = |W|$, the number of roots of unity contain in K

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  We write the fundamental system of units by $\epsilon_1, \dots \epsilon_d$. We have our embeddings $k \to k^{(i)}$ with $1 \le i \le r$.
  After order the embeddings by writing the real embeddings first, then write the complex embeddings, and then write the conjugates of the complex embeddings. We look at the determinant of a matrix involving the units and call that the regulator:

$$R = \det\left(\log|\epsilon_i^{(j)}|\right)_{1\le i,j\le r_0}$$

Intermezzo: Hurwitz zeta-function

Definition

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- Note that $\zeta(s,1) = \zeta(s)$, the Riemann L-function
- Hurwitz introduced this to study Dirichlet L-functions:

$$\begin{aligned} L(s,\chi) &= \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \\ &= \sum_{a=1}^{q} \chi(a) \sum_{n \equiv a \pmod{q}} \frac{1}{n^s} \\ &= \sum_{a=1}^{q} \frac{1}{q^s} \sum_{t=0}^{\infty} \frac{1}{(t+\frac{a}{q})^s} \end{aligned}$$

Hecke L-function (1/2)

Definition using ideals

Take an ideal $\mathfrak{f}$ of $\mathcal{O}_K$, and define the $\mathfrak{f}$-ideal class group as follows:
We say $\mathfrak{a} \sim_{\mathfrak{f}} \mathfrak{b}$ if there exists $\alpha, \beta \in \mathcal{O}_K$ such that $(\alpha)\mathfrak{a} = (\beta)\mathfrak{b}$, $\alpha \equiv \beta \pmod{\mathfrak{f}}$, and α/β is totally positive (all conjugates positive).

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This gives rise to a finite group, the $\mathfrak{f}$-ideal ray class group. Let $\mathcal{H}_{\mathfrak{f}}$ denote this group, and take a character $\chi : \mathcal{H}_{\mathfrak{f}} \to \mathbb{C}^\times$

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- Of course, we still have the notation of primitive character.

Hecke L-function (2/2)

Main Theorem of Class Field Theory

Given an algebric number field F and an ideal $\mathfrak{f}$ of $\mathcal{O}_F$, there exists a field extension $F_{\mathfrak{g}}/F$ whose Galois group is isomorphic to $\mathcal{H}_{\mathfrak{f}}$. Moreover, any finite abelian extension of F is contained in some $F_{\mathfrak{f}}$

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Hilbert's 12th problem

Extend the Kronecker-Weber Theorem on abelian extensions of the rational numbers to any base number field.

Artin L-function and Artin Map (1/4)

Preliminary Definitions

Let k be a number field, K be its finite Galois extension and $G = \mathrm{Gal}(K/k)$. Recall that we have the ring of integers downstairs $\mathcal{O}_k$ and upstairs $\mathcal{O}_K$ with $\mathcal{O}_k \subset \mathcal{O}_K$. Given $\mathfrak{p} \in \mathcal{O}_k$ we can write it upstairs: $\mathfrak{p}\mathcal{O}_K = \mathfrak{P}_1^{e_1} \ldots \mathfrak{P}_g^{e_g}$

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Apply a $\sigma \in G$. The left hand side stays the same, namely $\mathfrak{p}\mathcal{O}_K$, while the right hand side becomes $\sigma(\mathfrak{P}_1)^{e_1} \ldots \sigma(\mathfrak{P}_g)^{e_g}$. Thus G acts on $\{\mathfrak{P}_1 \ldots \mathfrak{P}_g\}$ transitively.

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Letting $D_{\mathfrak{P}}$ denote the decomposition group at P, namely $\{\sigma \in G : \sigma(\mathfrak{P}) = \mathfrak{P}\}$, we get the inertia group

$$I_{\mathfrak{P}} = \{\sigma \in G : \sigma(x) \equiv x \pmod{\mathfrak{P}} \text{ for all } x \in \mathcal{O}_K\}$$

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We have $I_{\mathfrak{P}}$ is a normal subgroup of $D_{\mathfrak{P}}$, and can study $(\mathcal{O}_K/\mathfrak{P})/(\mathcal{O}_k/\mathfrak{p})$ We get a canonical isomorphism:

$$D_{\mathfrak{P}}/I_{\mathfrak{P}} \sim \mathrm{Gal}((\mathcal{O}_K/\mathfrak{P})/(\mathcal{O}_k/\mathfrak{p}))$$

It is cyclic: $x \mapsto x^{N_{\mathfrak{p}}}, \sigma(x) \equiv x^{N\mathfrak{p}} \pmod{\mathfrak{P}}$. We have $\sigma_{\mathfrak{P}}$ well-defined up to $I_{\mathfrak{P}}$. The Frobenius automorphism at $\mathfrak{P}$. The $\sigma_{\mathfrak{P}_i}$'s are all conjugate. Call $\sigma_{\mathfrak{p}}$ the conjugacy class of $\sigma_g\mathfrak{p}$, the Artin symbol at $\mathfrak{p}$.

Artin L-function and Artin Map (2/3)

Definition of Artin L-function

Denote $\rho : G \to \mathrm{GL}(V)$ to be a finite dimensional representation

$$L(s, \rho, K/k) = \prod_{\mathfrak{P}} \det\left(1 - \rho(\sigma_{\mathfrak{P}}) N\mathfrak{P}^{-s} | V^{I_{\mathfrak{P}}}\right)^{-1}$$

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Artin's Reciprocity Law

If ρ is 1-dimensional then there exists an $\mathfrak{f} \in \mathcal{O}_k$ and a character χ of $\mathcal{H}_{\mathfrak{f}}$ such that

$$L(s,\rho,K/k) = L(s,\chi)$$

where $L(s,\chi)$ are the Hecke L-functions.

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where $L(s, \chi)$ are the Hecke L-functions.

Hecke generalized Riemann's results to abelian case, and proves Artin's conjecture in this case. If we analyze what the above is saying in the special case of a quadratic extension ($k = \mathbb{Q}$ and K quadratic), we get the (classical) quadratic reciprocity law.

Artin L-function and Artin Map (3/4)

Brauer's Induction Theorem

Given any character χ of a finite group G, there exist nilpotent subgroups H_i and ψ_i, one-dimensional characters of H_i such that

$$\chi = \sum_i a_i \mathrm{Ind}_H^G \psi_i$$

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Now we find

$$\begin{aligned} L(s,\chi,K/k) &= \prod L(s, \mathrm{Ind}_H^G \psi_i, K/k)^{a_i} \\ &= \prod_i L(s,\psi_i,K/K^{H_i})^{a_i} \\ &= \prod_i L(s,\phi_i)^{a_i} \quad \text{(Hecke character)} \end{aligned}$$

which gives a meromorphic continuation.

Artin L-function and Artin Map (4/4)

Artin Map

Let K be a number field, $C_K := \mathbb{A}_K^\times / K^\times$, the idèle class group of K. There is a homomorphism, called the Artin map,

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(i) For every finite abelian extension L/K, let $\theta_{L/K}$ denote the composition of

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Then $\ker \theta_{L/K} = N_L/K(C_L)$, which yields an isomorphism

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(ii) Given any open subgroup of N of C_K of finite index, there is a finite abelian extension L of K with $N = \ker \theta_{L/K}$.

Relation to Langlands Program

Artin's Conjecture

Let $\rho : \mathrm{Gal}(\bar{\mathrm{K}}/\mathrm{K}) \to \mathrm{GL_n}(\mathbb{C})$ be an irreducible non-trivial Galois representation. Then $L(s, \rho)$ is entire.

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Remarks

If ρ is trivial, then $L(s, \rho) = \zeta_K(s)$, the Dedekind L-function, with pole at $s = 1$.
If ρ is 1-dimensional, then $\rho = \chi$ corresponds to a non-trivial Hecke character, which is known to be entire.

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Langlands-Tunnell Theorem

Suppose $\rho : \mathrm{Gal}(L/K) \to \mathrm{GL}_2(\mathbb{C})$ is an irreducible 2-dimensional representation. If the image of ρ is solvable (a solvable subgroup of $\mathrm{GL}_2(\mathbb{C})$), then

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The above theorem was important in the proof of the Modularity Theorem!