Representations of the General Linear group of degree 2 over a Finite Field

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Introduction

Motivation

Although the title of this seminar "Representation Theory of $\mathrm{GL}(2, \mathbb{Q}_p)$" suggests that much we will be discussing here is the representations of $\mathrm{GL}(2)$, the 2-by-2 invertible matrices, over the p-adic field $\mathbb{Q}_p$, we take some time today to carefully examine the representation of $\mathrm{GL}(2)$ over finite fields first: We are able to later generalize (with modification) much of our theory to local fields and adele rings over a global field.

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Lecture Plan

We begin by defining a few important subgroups of $\mathrm{GL}(2, \mathbb{F}_q)$, classify $\mathrm{GL}(2, \mathbb{F}_q)$ by their conjugacy classes, then investigate when their induced representations are irreducible, and finally compute the character table.

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Conventions in This Talk

All representations here will be assumed finite-dimensional over the complex numbers. G will always assumed of finite order.

$\mathrm{GL}(2, \mathbb{F})$ with $\mathbb{F}$ arbitrary (1/2)

Important Algebraic Subgroups of $G_{\mathbb{F}} = \mathrm{GL}(2, \mathbb{F})$

The standard Borel subgroup B of $G_{\mathbb{F}}$, and the unipotetnt radical N of B:

$$N := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in G_{\mathbb{F}} \right\} \subset B := \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in G_{\mathbb{F}} \right\}$$

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i.e. $\{1, w_0\}$ is a set of representatives of coset space $B \backslash G / B$.

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Conjugacy Classes of $G_{\mathbb{F}} = \mathrm{GL}(2, \mathbb{F})$

Let $g \in G_{\mathbb{F}}$ and set $f(t) = \det(tI - g)$, the characteristic polynomial of g. Three different cases arise depending on the roots of $f(t)$:

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(1) $f(t)$ is irreducible over $\mathbb{F}$, and if $f(t) = t^2 + at + b$, then g is G-conjugate to:

$$\begin{pmatrix} 0 & -b \\ 1 & -a \end{pmatrix}$$

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(3) $f(t)$ has repeated roots $a \in \mathbb{F}^{\times}$, then g is conjugate to exactly one of:

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$$

Facts About $G = \mathrm{GL}_2(\mathbb{F}_q)$

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- Representations with the same character are isomorphic.
- The number of irreducible representations of G coincides with the number of its conjugacy classes. $\implies G = \mathrm{GL}_2(\mathbb{F}_q)$ ought to have exactly $q^2 - 1$ irreducible representations.

Induced Representations (1/2)

Definitions (Restriction v.s. Induced Representations)

Let $H \subset G$ be a subgroup, any representation V of G restricts to a representation of H, is called a restricted representation, denoted by $\mathrm{Res}_H^G(V)$ or simply $\mathrm{Res}(V)$

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Now suppose V is a representation of G, $W \subset V$ a subspace that is H-invariant. For any $g \in G$, the subspace $g \cdot W = \{g \cdot w | w \in W\}$ depends only on the left coset gH of g modulo H, $gh \cdot W = g \cdot (h \cdot W) = g \cdot W$; for a coset σ in G/H, we write $\sigma \cdot W$ for this subspace of V.

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We say that V is induced by W if every element in V can be written UNIQUELY as a sum of elements in such translates of W,

$$\text{i.e. } V = \bigoplus_{\sigma \in G/H} \sigma \cdot W$$

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$\mathrm{Ind}_H^G(W) := \{f : G \to W | f(hg) = \sigma(h) f(g) \text{ for all } h \in H, g \in G\}$

Induced Representations (2/2)

Frobenious Reciprocity Law

Res and Ind are adjoint functors between the category of G–modules and the category of H–modules:

$$\mathrm{Hom}_H(W, \mathrm{Res}(V)) = \mathrm{Hom}_G(\mathrm{Ind}(W), V),$$

or, equivalently

$$V \otimes \mathrm{Ind}(W) = \mathrm{Ind}(\mathrm{Res}(V) \otimes W).$$

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Theorem from Mackey Theory

Let G be a finite group, H its subgroup. If σ is a representation of H and π an irreducible representation of G whose restriction contains σ, then the restriction of π to H is the direct sum over $H\backslash G$ of the conjugates σ^g, each with the same multiplicity.

Obvious Irreducible Representations of $G = \mathrm{GL}_2(\mathbb{F}_q)$ (1/2)

Forbenius Formula for Characters

Let V be a finite-dimensional representation of a finite group G, and let W be a representation of a subgroup $H \subset G$. Then the characters of V and W satisfy the inner product relation

$$\left\langle \chi_{\mathrm{Ind}(W)}, \chi_V \right\rangle = \left\langle \chi_W, \chi_{\mathrm{Res}(V)} \right\rangle$$

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Straight Forward Construction of Irreducible Characters (1/2)

First consider the permutation representation of G on $\mathbb{P}^1(\mathbb{F}_q)$, which has dimension $q+1$, it contains the trivial representations; let V be the complementary q-dimensional representation.

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$$\chi(a_x) = q,\ \chi(b_x) = 0,\ \chi(c_{x,y}) = 1,\ \chi(d_{x,y}) = -1$$

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Since $\langle \chi, \chi \rangle = 1$, V is irreducible (another fact from prerequisite)

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Remark

The total number of irreducible representations contributed by U_α and V_α is $2q$, with q irreducible representations of each.

Definition and the Importance of Studying $\mathrm{Ind}_B^G \chi$ (1/2)

Constructions of Irreducible Representations of B

Let χ_1, χ_2 be characters of $\mathbb{F}_q^\times$, then we form the character χ of T:

$$\chi = \chi_1 \otimes \chi_2 : \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto \chi_1(a)\chi_2(b)$$

We regard this as a character of B, trivial on N, via the quotient $B \to B/N \cong T$

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Hence we form the induced representation $\mathrm{Ind}_B^G(\chi)$ of G, and by considering its irreducible component, we come at Proposition 1:

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Definition and the Importance of Studying $\mathrm{Ind}_B^G\chi$ (1/2)

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Outline of Proof to Proposition 1

The representations π contains the trivial character of N if and only if it contains an irreducible representation π of B containing the trivial character of N.

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How to approach $\mathrm{Ind}_B^G \chi$

We have to analyze the induced representation $\mathrm{Ind}_B^G \chi$; to do so, we form the character χ^w, of T: (χ^w is called the Weil conjugate of χ)

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We abuse notation here and view this as a character of B trivial on N.

Irreducibility of $\mathrm{Ind}_B^G \chi$ (1/3)

Proposition 2

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Let χ be a character of T, viewed as a character of B which is trivial on N:

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- The representation $\mathrm{Ind}_B^G \chi$ is IRREDUCIBLE if and only if $\chi \neq \chi^w$
- If $\chi = \chi^w$, the representation $\mathrm{Ind}_H^G \chi$ has length 2, with distinct decomposition factors.

Irreducibility of $\mathrm{Ind}_B^G \chi$ (2/3)

Proof of Proposition 2 (1/2)

The canonical isomorphism of Frobenius Reciprocity gives:

$$\mathrm{Hom}_G(\mathrm{Ind}_B^G(\chi), \mathrm{Ind}_B^G(\xi)) \cong \mathrm{Hom}_B(\mathrm{Res}_B^G(\mathrm{Ind}_B^G(\chi)), \xi) \tag{1}$$

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where $\chi^y(b) := \chi(yby^{-1})$ for any $b \in B^y = y^{-1}By$.

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where $\chi^y(b) := \chi(yby^{-1})$ for any $b \in B^y = y^{-1}By$.

By Bruchat decomposition, we only have to consider $y = 1$ and $y = w_0$:

- The term in (2) corresponding to $y = 1$ is just χ, and hence contributes a factor $\mathrm{Hom}_T(\chi, \xi)$ to (1)
- The term corresponding to w_0 contributes $\mathrm{Hom}_B(\mathrm{Ind}_T^B(\chi^w), \xi) \cong \mathrm{Hom}_T(\chi^w, \xi)$

Irreducibility of $\mathrm{Ind}_B^G\chi$ (3/3)

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Altogether, (1) decomposes to:

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Remark

The representation $\mathrm{Ind}_B^G \chi$ has dimension $[G : B] = q + 1$, with characters:

$$\begin{array}{ccccc} & a_x & b_x & c_{x,y} & d_{x,y} \\ W_{\alpha,\beta} & (q+1)\alpha(x)\beta(x) & \alpha(x)\beta(x) & \alpha(x)\beta(y) + \alpha(y)\beta(x) & 0 \end{array}$$

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The above implies $W_{\alpha,\beta} \cong W_{\beta,\alpha}$, and $W_{\alpha,\alpha} \cong U_\alpha \oplus V_\alpha$. Hence we just proved $W_{\alpha,\beta}$ is irreducible if $\alpha \neq \beta$, this gives $\frac{1}{2}(q-1)(q-2)$ more representations.

Missing Representations and the Rescue(r)

Character Table for $G = \mathrm{GL}_2(\mathbb{F}_q)$ so far

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Comparing with the list of conjugacy classes, we see that $\frac{1}{2}q(q-1)$ representations are missing. Natural ways to find new characters by inducing characters from the cyclic subgroup of G FAIL to give irreducible characters (to be checked privately at home).

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X_ϕ	?	?	?	?

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Solution to this problem: Weil Representations (Next Week)