# The Weil Representations for the General Linear group of degree 2 over a Finite Field

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# Introduction

## Motivation

Recall last week we attempted (in vain) to compute the character table of their irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$, but a large portion of irreducible representations are missing, with our technology at the time. Today we shall "recover" the missing representations by a method André Weil developed in his celebrated paper *"Sur certains groupes d'opèrateurs unitaires"* dated 1964.

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## Lecture Plan

We first carefully scrutinize the quadratic extension of $\mathbb{F}_q$, then show they correspond to the generators of the group $\mathrm{SL}_2(\mathbb{F}_q)$, which representations can be extended to those of $\mathrm{GL}_2(\mathbb{F}_q)$

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## Reminder

Remember there were $q(q-1)$ representations missing to be found!

# Where We Left Off

Conjugacy Classes of $\mathrm{GL}_2(\mathbb{F}_q)$

| Representative | No. of Elements in Class | No. of Classes |
|---|---|---|
| $a_x = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ | $1$ | $q-1$ |
| $b_x = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$ | $q^2-1$ | $q-1$ |
| $c_{x,y} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ | $q^2+q$ | $\frac{(q-1)(q-2)}{2}$ |
| $d_{x,y} = \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}$ | $q^2-q$ | $\frac{q(q-1)}{2}$ |

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Character Table

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| $G$ | $a_x = \left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$ | $b_x = \left(\begin{smallmatrix} x & 1 \\ 0 & x \end{smallmatrix}\right)$ | $c_{x,y} = \left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$ | $d_{x,y} = \left(\begin{smallmatrix} x & \varepsilon y \\ y & x \end{smallmatrix}\right) = \zeta$ |
| $U_\alpha$ | $\alpha(x^2)$ | $\alpha(x^2)$ | $\alpha(xy)$ | $\alpha(\zeta^q)$ |
| $V_\alpha$ | $q\alpha(x^2)$ | $0$ | $\alpha(xy)$ | $-\alpha(\zeta^q)$ |
| $W_{\alpha,\beta}$ | $(q+1)\alpha(x)\beta(x)$ | $\alpha(x)\beta(x)$ | $\alpha(x)\beta(y)+\alpha(y)\beta(x)$ | $0$ |
| $X_\phi$ | ? | ? | ? | ? |

# Quadratic Extensions of $\mathbb{F}_q$ (1/2)

## Observations

Notice the conjugacy class with representative $d_{x,y} \in \mathrm{GL}_2(\mathbb{F}_q) = G$, with $\varepsilon$ non-square in $\mathbb{F}_q$, gives an interesting isomorphism

$$K := \left\{ \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} \right\} \cong \mathbb{F}_{q^2}^\times, \quad \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} \leftrightarrow \zeta = x + y\sqrt{\varepsilon}$$

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## Failed Irreducible Representations (1/2)

A natural way to find new characters of $G$ is to *induce* characters from $K$: For a representation $\phi : K \to \mathbb{C}^{\times}$ the character values of induced (from $K$ to $G$, denoted $\mathrm{Ind}(\phi)$) representation are:

$a_x \mapsto q(q-1)\phi(x) \quad b_x \mapsto 0 \quad c_{x,y} \mapsto 0 \quad d_{x,y} \mapsto \phi(\zeta) + \phi(\zeta)^q$

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Note that $\mathrm{Ind}(\phi) \cong \mathrm{Ind}(\phi^q)$, so it gives $\frac{1}{2}q(q-1)$ different representations when $\phi \neq \phi^q$ (characters with this property are called regular).

# Quadratic Extensions of $\mathbb{F}_q$ (2/2)

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However, these representations are NOT irreducible in the following sense: the character $\chi$ of $\mathrm{Ind}(\phi)$ satisfies

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A irreducible representation of $G$ NOT containing the trivial character of $N$ is called the *cuspidal*. Such representations $\sigma$ contain all non-trivial characters of $N$.

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Thus the above failed example tells us that cuspidal representations cannot be constructed directly from induction.

# $\mathbb{F}_{q^2}$ and $\mathrm{SL}_2(\mathbb{F}_q)$

## Generators for $\mathrm{SL}_2(\mathbb{F}_q)$

Let $S$ be the group generated by $t(y)$, $n(z)$, and $w_1$, with $y \in \mathbb{F}_q^\times$, $z \in \mathbb{F}_q$, where functions $t$ and $n$ satisfy:

$$t(y_1)t(y_2) = t(y_1 y_2); \quad n(z_1)n(z_2) = n(z_1 + z_2); \tag{1}$$

$$t(y)n(z)t(y)^{-1} = n(y^2 z); \quad wt(y)w = t(-y^{-1}); \tag{2}$$

$$wn(z)w = t(-z^{-1})n(-z)wn(-z^{-1}), \quad (z \neq 0) \tag{3}$$

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Then $S$ is isomorphic to $\mathrm{SL}_2(\mathbb{F}_q)$: in this isomorphism

$$t(y) \mapsto \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}, \; n(z) \mapsto \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, \; w \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

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$\psi : \mathrm{SL}_2(\mathbb{F}_q) \to S$ by $\psi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} n(a/c)t(-c^{-1})wn(d/c) & \text{if } c \neq 0 \\ t(a)n(b/a) & \text{if } c = 0 \end{cases}$

# The space $W$

Let $E$ be a two-dimensional commutative semi-simple algebra over $\mathbb{F}_q$, then precisely two non-isomorphic possibility for E:

- split case: $E = \mathbb{F}_q \oplus \mathbb{F}_q$ with $F$ embedded diagonally
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Let $x \to \bar{x}$ be the automorphism $(\xi, \eta) \to (\eta, \xi)$ of $E$ if $E$ splits, or let $x \to \bar{x}$ be the nontrivial Galois automorphism of $E/\mathbb{F}_q$ if $E$ is anisotropic.

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Define $\mathrm{tr}, \mathrm{Norm} : E \to \mathbb{F}_q$ be the trace and Norm maps ($\mathrm{tr}(x) := x + \bar{x}$, $\mathrm{Norm}(x) := x\bar{x}$)
If $\Phi$ is a function on $E$ we define the *Fourier transform* $\hat{\Phi}$ by

$$\hat{\Phi}(x) := \epsilon \frac{1}{q} \sum_{y \in E} \Phi(y) \psi(\mathrm{tr}(\bar{x}y)), \qquad \epsilon := \begin{cases} 1 & \text{in the split case,} \\ -1 & \text{in the anisotropic case} \end{cases}$$

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Let $W$ be the $q^2$-dimensional space of all complex-valued functions on $E$

# Weil Representation for $\mathrm{SL}_2(\mathbb{F}_q)$ (1/2)

There exists a representation $\omega : \mathrm{SL}_2(\mathbb{F}_q) \to \mathrm{End}(W)$ such that

$$\left(\omega\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\Phi\right)(x) = \Phi(yx),$$
$$\left(\omega\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\Phi\right)(x) = \psi(z\mathrm{Norm}(x))\Phi(x),$$
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## Remark

If $y = 1$, then the second formula in (2) is the Fourier inversion formula $\hat{\hat{\Phi}}(x) = \Phi(-x)$.

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We need to verify the consistency relations corresponding to (1)-(3) above. We outline how (3) can be obtained:

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Need to show: $\omega\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\omega\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\omega\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
$= \omega\begin{pmatrix} -a^{-1} & 0 \\ 0 & -a \end{pmatrix}\omega\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}\omega\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\omega\begin{pmatrix} 1 & -a^{-1} \\ 0 & 1 \end{pmatrix}$

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With $a \in \mathbb{F}_q^\times$ and $b \in E$ we can deduce:

$$\sum_{y \in E} \psi(a\mathrm{Norm}(y) + \mathrm{tr}(\bar{b}y)) = \epsilon q \psi(-a^{-1}\mathrm{Norm}(b))$$

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Unfolding $L :=$ (L.H.S. $\Phi)(x)$ and $R :=$ (R.H.S. $\Phi)(x)$ we see:

$$L = \frac{1}{q^2} \sum_{y,z \in E} \psi(a\mathrm{Norm}(y))\psi(\mathrm{tr}(\bar{z}y))\Phi(z)$$

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# Extending Representations from $\mathrm{SL}_2(\mathbb{F}_q)$ to $\mathrm{GL}_2(\mathbb{F}_q)$ (1/2)

## Representations of $E^\times$

Let $\chi$ be a character of $E^\times$. We will associate a representation $(\pi(\chi), W(\chi))$ of $\mathrm{GL}_2(\mathbb{F}_q)$ with $\chi$. We assume that $\chi$ does NOT factor through $\mathrm{Norm} : E^\times \to \mathbb{F}_q^\times$. Let $E_1^\times$ be the subgroup of elements $x$ with $\mathrm{Norm}(x) = 1$

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Let

$$W(\chi) := \{\Phi \in W : \Phi(yx) = \chi(y)^{-1}\Phi(x) \text{ for } y \in E_1^\times\}$$

$$\dim(W(\chi)) = \begin{cases} q-1 \text{ if } E \text{ splits} \\ q+1 \text{ if } E \text{ is anisotropic} \end{cases}$$

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## Remark

Our assumption is equivalent to saying that $\chi|E_1^\times$ is nontrivial. Without the assumption we still could construct $\pi(\chi)$, but NOT irreducible.

# Extending Representations from $\mathrm{SL}_2(\mathbb{F}_q)$ to $\mathrm{GL}_2(\mathbb{F}_q)$ (2/2)

Now extend the action of $\mathrm{SL}_2(\mathbb{F}_q)$ on $W(\chi)$ to a representation of $\mathrm{GL}_2(F_q)$ by letting

$$\left(\omega\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\Phi\right)(x) := \chi(b)\Phi(bx)$$

where $a \in \mathbb{F}_q^\times, b \in E^\times$ are chosen so that $\mathrm{Norm}(b) = a$.

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## Weil Representations are cuspidal

Finally we can show (NOT today) that the Weil Representations $(\pi(\chi), W(\chi))$ are cuspidal. (i.e. there exists NO nonzero linear functional $l$ on $W(\chi)$ such that

$$l\left(\pi\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}v\right) = l(v), \quad \forall v \in V, x \in \mathbb{F}_q.)$$

Note we just need to check the case when $E$ anisotropic.

# The Completed Character Table

Weil Representations provide us the virtual characters:

$$\chi_\phi := \chi_{V \otimes W_{a,1}} - \chi_{W_{a,1}} - \chi_{\mathrm{Ind}(\phi)}$$

where $V, W_{a,b}$ were defined last time and recall $\phi : K \cong \mathbb{F}_{q^2} \to \mathbb{C}^\times$

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| $V_\alpha$ | $q\alpha(x^2)$ | $0$ | $\alpha(xy)$ | $-\alpha(\zeta^q)$ |
| $W_{\alpha,\beta}$ | $(q+1)\alpha(x)\beta(x)$ | $\alpha(x)\beta(x)$ | $\alpha(x)\beta(y) + \alpha(y)\beta(x)$ | $0$ |
| $X_\phi$ | $(q-1)\phi(x)$ | $-\phi(x)$ | 0 | $-(\phi(\zeta) + \phi(\zeta^q))$ |