What is the Riemann Hypothesis and why do we care?

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The University of British Columbia

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Discrete v.s. Continuous

A Major task of mathematics today is to harmonize the continuous and the discrete, to include them in one comprehensive mathematics, and to eliminate obscurity from both.

— E. T. Bell, “Men of Mathematics”, 1937

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- Chebyshev (1850): There is always a prime p such that $n < p < 2n$ for $n \in \mathbb{Z}_{>1}$.
- Gauss (Christmas Eve 1849): As a boy of 15 or 16, I determined that, at around x, the primes occur with density $\frac{1}{\log x}$, i.e.,

$$\pi(x) := \#\{\text{primes } \leq x\} \approx \sum_{n=2}^{\lfloor x \rfloor} \frac{1}{\log n} \approx \int_2^x \frac{dt}{\log t} := \mathrm{Li}(x)$$

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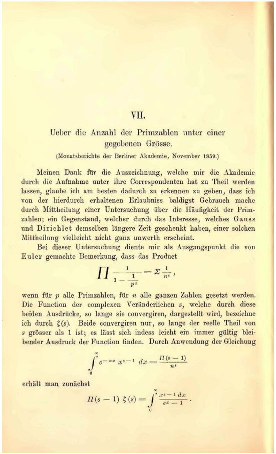

VII.

Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

(Monatsberichte der Berliner Akademie, November 1859.)

Meinen Dank für die Auszeichnung, welche mir die Akademie durch die Aufnahme unter ihre Correspondenten hat zu Theil werden lassen, glaube ich am besten dadurch zu erkennen zu geben, dass ich von der hierdurch erhaltenen Erlaubniss baldigst Gebrauch mache durch Mittheilung einer Untersuchung über die Häufigkeit der Primzahlen; ein Gegenstand, welcher durch das Interesse, welches Gauss und Dirichlet demselben längere Zeit geschenkt haben, einer solchen Mittheilung vielleicht nicht ganz unwerth erscheint.

Bei dieser Untersuchung diente mir als Ausgangspunkt die von Euler gemachte Bemerkung, dass das Product

$$\Pi \frac{1}{1-\frac{1}{p^s}} = \Sigma \frac{1}{n^s},$$

wenn für p alle Primzahlen, für n alle ganzen Zahlen gesetzt werden. Die Function der complexen Veränderlichen s, welche durch diese beiden Ausdrücke, so lange sie convergiren, dargestellt wird, bezeichne ich durch $\zeta(s)$. Beide convergiren nur, so lange der reelle Theil von s grösser als 1 ist; es lässt sich indess leicht ein immer gültig bleibender Ausdruck der Function finden. Durch Anwendung der Gleichung

$$\int_0^\infty e^{-nx} x^{s-1}\, dx = \frac{\Pi(s-1)}{n^s}$$

erhält man zunächst

$$\Pi(s-1)\, \zeta(s) = \int_0^\infty \frac{x^{s-1}\, dx}{e^x - 1}.$$

Riemann's 1859 memoir: the main contributions

(1) Riemann considered $\zeta(s)$ by taking $s \in \mathbb{C}$, analytically continued it to the entire complex plane, completed it with the Gamma factor

$$\xi(s) := \frac{1}{2}\pi^{-s/2}s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

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(2) He also introduced the discrete function $J(x)$

$$J(x) := \sum_{n \leq x} \frac{\pi(x^{1/n})}{n} = \pi(x) + \frac{\pi(x^{1/2})}{2} + \frac{\pi(x^{1/3})}{3} + \cdots,$$

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$$\log\zeta(s) = \int_0^\infty x^{-s}\,\mathrm{d}J(x) = s\int_0^\infty J(x)x^{-s-1}\,\mathrm{d}x.$$

The zeros of the Riemann ζ-function

The two types zeros of $\zeta(s)$

We see that the completed ξ-function

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- As the Gamma factor $\Gamma\left(\frac{s}{2}\right)$ has simple poles at $s = 0, -2, -4, -6$, ..., the zeta function has simple zeros at $s = -2, -4, -6, \ldots$, these are the *trivial zeros*.

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The Riemann Hypothesis

All the non-trivial zeros of $\zeta(s)$ lie on the vertical line $s = 1/2$.

To rescue $\pi(x)$

Now after we attempt to recover $\pi(x)$ from the $J(x)$:

$$\pi_0(x) := \frac{1}{2}\lim_{h\to 0}\left(\pi(x+h)+\pi(x-h)\right) = \sum_{n\geq 1}\mu(n)\frac{J(x^{1/n})}{n},$$

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where $\mu(n)$ is the Möbius μ-function

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ has one or more repeated prime factors} \\ 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes.} \end{cases}$$

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$$J(x) = \mathrm{li}(x) - \sum_{\substack{\rho \\ \text{non-trival zeros of } \zeta(s)}} \mathrm{li}(x^\rho) - \log(2) + \int_x^\infty \frac{dt}{t(t^2-1)\log t}.$$

The Prime Number Theorem and the zero free region

Two French mathematicians, J. Hadamard and C. de la Vallée-Poussin in 1896 independently proved that the zeta function has no non-trivial zeros in the region

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Thus proved the celebrated

Prime Number Theorem (1859)

$$\pi(x) \sim \frac{x}{\log x},$$

which was conjectured by Gauss and Legendre.

Zeros on the critical line $\rho = 1/2$:

Since the Riemann Hypothesis asserts that all the non-trivial zeros lie on the vertical line $\rho = 1/2$, there has been numerous evidence (attempt to) support it:

Timeline of the results on the non-trivial zeros being critical

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- B. Conrey (1989) improved the above to more than 40.7% ("at least $2/5$") of the zeros of $\zeta(s)$ are simple and lie on the critical line.

Other consequences of the validity Riemann Hypothesis

Gaps between primes

H. Cramér in 1937 proved that under RH,

$$p_{n+1} - p_n = o(\log p_n)^3,$$

where p_n denotes the nth prime number.

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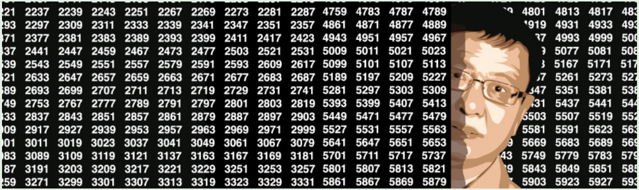
–Update: Y. Zhang in May 2013 gave the unconditional bound

$$\liminf_{n\to\infty} (p_{n+1} - p_n) < 7 \times 10^7.$$



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Further consequences:

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