Basics of Hodge-de Rham Theory I: Motivations and Definitions

Justin Scarfy

The University of British Columbia

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Introduction and Plan

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Plan and Warnings for this lecture

This lecture will try to drown you with definitions: we start with a quick review of **de Rham theory**, unify certain aspects between **Ćech cohomology** and **singular cohomology**, survey a few facts about **Kähler manifolds**, define **resolutions** and **hypercohomology**, mention a few well known **cohomological realizations** of complex projective varieties, and define many different **Hodge structures** to play with in the upcoming

(Review of) de Rham Theory (1/3)

Let $x_1, \ldots, x_n$ be the standard coordinates in $\mathbb{R}^n$, recall that the de Rham complex of $\mathbb{R}^n$, Ω^*, is defined to be the free $\mathbb{R}$-algebra (with units) generated by symbols $dx_1, \ldots, dx_n$, with the relations

$$\begin{cases} (dx_i)^2 = 0, \\ dx_i dx_j = -dx_j dx_i. \end{cases}$$

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i.e., $\omega \in \Omega^n(\mathbb{R}^n)$ iff

$$\omega = \sum_I f_I \, dx_I,$$

with the multi-index notation $I := (i_1, \ldots, i_g)$, $dx_I := dx_{i_1} \cdots dx_{i_g}$.

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3. For $\alpha \in \Omega^r(\mathbb{R}^n)$, $\beta \in \Omega^s(\mathbb{R}^n)$, the **Leibniz rule** holds:
$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^r \alpha \wedge d\beta.$$

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Example with $q = 3$, $n = 3$

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\Omega^0(\mathbb{R}^3) \arrow[r, "d"] \arrow[d, Rightarrow] & \Omega^1(\mathbb{R}^3) \arrow[r, "d"] \arrow[d, Rightarrow] & \Omega^2(\mathbb{R}^3) \arrow[r, "d"] \arrow[d, Rightarrow] & \Omega^3(\mathbb{R}^3) \arrow[d, Rightarrow] \\
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\{\text{functions}\} \arrow[r, "\mathrm{grad}"] & \{\text{vector fields}\} \arrow[r, "\mathrm{curl}"] & \{\text{vector fields}\} \arrow[r, "\mathrm{div}"] & \{\text{functions}\}
\end{tikzcd}$$

(Review of) de Rham Theory (3/3)

de Rham (Dolbeault) cohomology

Let $M \subset \mathbb{R}^n$ be a smooth manifold, we define

$$H^q_{dR}(M;\mathbb{R}) := \frac{\ker\{d : \Omega^q(M) \to \Omega^{q+1}(M)\}}{\operatorname{im}\{d : \Omega^{q-1}(M) \to \Omega^q(M)\}} = \frac{\{\text{closed } q\text{-forms in } M\}}{\{\text{exact } q\text{-forms in } M\}}.$$

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de Rham Theorem

The theorem of de Rham (1931) asserts that for a smooth manifold $M \subset \mathbb{R}^n$,

$$H^q_{dR}(M,\mathbb{R}) \cong H_{\text{sing}}(M,\mathbb{R}),$$

the singular cohomology of M with coefficients in $\mathbb{R}$.

Complex of Sheaves

Example 1 of a sheaf: Verify this yourself!

For a smooth manifold M, the set of differential p-forms Ω^p_M **form a sheaf** that assigns to each open set U of M the set $\Omega^p_M(U)$ of smooth p-forms on U.

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$\mathbb{R}_M$, the locally constant sheaf on M that assigns to each open set U the set $\mathbb{R}_M(U)$ of locally constant function on U.

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Complex of Sheaves

A **complex of sheaves** is a collection of sheaves $\mathcal{F}_i, i \in \mathbb{Z}$, together with morphisms of sheaves $d_i : \mathcal{F}_i \to \mathcal{F}_{i+1}$ such that $d_{i+1} \circ d_i = 0$.

Resolutions

Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be three sheaves, and let $\phi : \mathcal{F} \to \mathcal{G}, \psi : \mathcal{G} \to \mathcal{H}$ be morphisms of sheaves such that $\psi \circ \phi = 0$.

Resolutions

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The complex $\mathcal{F}^\bullet$ is called a **resolution** of $\mathcal{F}$ if for every $i \geq 0$, the sequence

$$\mathcal{F}^i \xrightarrow{\phi_i} \mathcal{F}^{i+1} \xrightarrow{\phi_{i+1}} \mathcal{F}^{i+2}$$

is exact in the middle, and j is injective with $j(\mathcal{F}) = \ker \phi_0$.

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Fine resolution

The complex $\mathcal{F}^\bullet$ is called a **fine resolution** of $\mathcal{F}$ if for every $i \geq 0$, the sequence

$$0 \to \mathcal{F} \to \mathcal{F}^0 \xrightarrow{\phi_0} \mathcal{F}^1 \xrightarrow{\phi_1} \mathcal{F}^2 \xrightarrow{\phi_2} \cdots$$

is everywhere exact.

Čech cohomology (1/3)

Let $\mathcal{I}$ be an totally ordered set, and define an **abstract simplicial complex** Δ on $\mathcal{I}$ be a collection of a finite subset of $\mathcal{I}$, closed under taking subsets. Each $F \in \Delta$ is called a **face** of Δ. If $\mathcal{U} = \{U_i\}_{i \in I}$ is a locally finite cover of M, then we associate to $\mathcal{U}$ an abstract simplicial complex $\mathcal{N}(U)$, called the **nerve** of U, and the faces of $\mathcal{N}(U)$ are the sets with $\{U_{i_1}, \ldots, U_{i_k}\}$ with $U_{i_1} \cap \cdots \cap U_{i_k} \neq \emptyset$.

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Refinement

Let $\mathcal{U}$ and $\mathcal{V}$ be open covers of M, then $\mathcal{V}$ is a **refinement** of $\mathcal{U}$ if every element in $\mathcal{V}$ is contained in $\mathcal{U}$.

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Čech Theorem

Let $\mathcal{U}$ be a locally finite open cover of M, and if all intersections of sets in $\mathcal{U}$ (including $\mathcal{U}$) are contractible, then $\mathcal{N}(U)$ is homotopically equivalent to M.

Čech cohomology (2/3)

Čech covers

A cover which satisfies the hypothesis in the above Theorem is called a Čech cover, and all smooth manifolds admit Čech covers (fact).

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Čech cochains

Let $\mathcal{F}$ be a sheaf on M, and $\mathcal{U} = \{U_i\}$ be a locally finite open cover of M. A **Čech** k**-cochain** is a function α on the k-faces of $\mathcal{N}(U)$ such that the value on the face $U_{i_0}, \ldots, U_{i_k}$ lies in $\mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_k})$, i.e., the group of k-cochains is

$$\check{C}^k(\mathcal{U}, \mathcal{F}) = \bigoplus_{i_0 < \cdots < i_k} (U_{i_0} \cap \cdots U_{i_k}).$$

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$$\check{C}^k(\mathcal{U}, \mathcal{F}) = \bigoplus_{i_0 < \cdots < i_k} (U_{i_0} \cap \cdots U_{i_k}).$$

with the coboundary operator d sends k-cochains to $(k+1)$-cochains:

$$d : \alpha(U_{i_0} \cap \cdots \cap U_{i_k}) \mapsto \sum_{j=0}^{k+1} (-1)^j \alpha(U_{i_0} \cap \cdots \cap \widehat{U}_{i_j} \cap \cdots \cap U_{i_k}).$$

Čech cohomology (3/3)

Since the coboundary operator d satisfies $d^2 = 0$, we obtain a chain complex of chains $\mathcal{F}^\bullet(X)$ of chains, which we call the **Čech complex** of $\mathcal{F}$ associated to $\mathcal{U}$, and the **Čech cohomology** of $\mathcal{F}$ with respect to $\mathcal{U}$ is defined to be:

$$\check{H}^k(\mathcal{U}, \mathcal{F}) := \frac{\ker d : \mathcal{F}^k(M) \to \mathcal{F}^{k+1}(M)}{\operatorname{im} d : \mathcal{F}^{k-1}(M) \to \mathcal{F}^k(M)}$$

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Čech cohomology group

Let $\mathcal{F}$ be a sheaf of M, the kth Čech cohomology group of $\mathcal{F}$ is

$$\check{H}^k(M, \mathcal{F}) = \varinjlim_{\mathcal{U}} \check{H}^k(\mathcal{U}, \mathcal{F})$$

where the covers $\mathcal{U}$ are ordered by refinement.

Proof of de Rham Theorem

It can be shown that $\check{H}^k(M, \mathbb{R}_M)$ and $H^k_{\text{sing}}(M, \mathbb{R})$ are isomorphic, as singular cochains can be approximated by Čech cochains for a very refined cover $\mathcal{U}$ of M.

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Let $\mathcal{F}$ be a sheaf on M and let

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots$$

be a **fine resolution** of $\mathcal{F}$. Suppose that $\mathcal{U}$ is an open cover of M such that the sequence of homomorphisms

$$\mathcal{F}^{j-1}(U_{i_0} \cap \cdots \cap U_{i_k}) \to \mathcal{F}^j(U_{i_0} \cap \cdots \cap U_{i_k}) \to \mathcal{F}^{j+1}(U_{i_0} \cap \cdots \cap U_{i_k})$$

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$$\check{H}(\mathcal{U}, \mathcal{F}) \cong \check{H}(M, \mathcal{F}), \quad \text{and finally,} \quad \check{H}(M, \mathbb{R}_M) \cong H^k_{dR}(M).$$

□

Kähler manifolds

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Symplectic Structure

A **symplectic structure** on a $2d$-dimensional manifold M is a closed 2-form $\omega \in \bigwedge^2(M)$, the set of alternating 2-forms on M, such that $\Omega = \omega^d/d!$ (the volume form), is nowhere vanishing.
Reality test: What about when $\omega \in \mathrm{Sym}^2(M)$?

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Hermitian metric

Let $M \subset \mathbb{C}^n$ be a complex manifold and J is complex structure (atlas), and a Riemannian metric g on M is said to be a **Hermitian metric** if and only if for every $p \in M$, the bilinear form g_p on the tangent space $T_p(M)$, is compatible with the complex structure J_p.

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A Hermitian metric on a manifold M is said to be a **Kähler metric** if and only if the 2-form

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Hodge decomposition theorem

Let M be a compact Kähler manifold and $H^{p,q}(M)$ the space of de Rham cohomology classes for $H^{p+q}(M, \mathbb{C})$ that have a representation of bidegree (p, q). Then

$$H^k(M, \mathbb{C}) \cong \bigoplus_{p+q} H^{p,q}(M),$$
$$H^{p,q} = \overline{H^{q,p}}.$$

Hypercohomology (1/3)

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$$0 \to I \to M \to N \to 0$$

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  splits.
- $\mathcal{F}$ is said to be **left-exact**, if for every short exact sequence $0 \to A \to B \to C \to 0$ of objects in $\mathcal{A}$, the sequence

  $$0 \to \mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(C)$$

  of objects in $\mathcal{B}$ is exact.

Hypercohomology (2/3)

Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories, where $\mathcal{A}$ has **sufficiently many injective objects** and let

$$\mathcal{F} : \mathcal{A} \to \mathcal{B}$$

be **left exact** functor. For a left-bounded complex $M^\bullet$ of $\mathcal{A}$, we define the ith derived object $R^i\mathcal{F}(M^\bullet)$ as follows:

Hypercohomology (2/3)

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Define the **hypercohomology**

$$\mathbb{H}^i(\mathcal{F}(I^\bullet)) := R^i\mathcal{F}(M^\bullet), \quad \text{the right derived functor of} \mathcal{F}.$$

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The hypercohomology $\mathbb{H}(X, \mathcal{L}^\bullet)$ is the total cohomology of the double complex, i.e., the cohomology of the associated single complex

$$K^\bullet = \bigoplus K^k = \bigoplus_k \bigoplus_{p+q=k} K^{p,q}$$

with differential $D := \delta + (-1)^p d$, (δ and d the horizontal and vertical differential of the complex K):

$$\mathbb{H}(X, \mathcal{L}^\bullet) := H^k_D(K^\bullet).$$

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Let k be a field with algebraic closure $\bar{k} \subseteq \mathbb{C}$ (so $\mathrm{char}(k) = 0$). Consider smooth projective varieties (so it a pure motive) X over k. There are several 'nice' cohomology theory:

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alternate definition of HS via (induced) filtration (1/2)

A HS of weight k consists of a real vector space V and a decreasing filtration

$$\cdots \subset F^p \subset F^{p-1} \subset \cdots$$

Hodge Structures

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Example: The Tate-Hodge structure (THS)

$\mathbb{Z}(1)$ is an HS of weight -2 defined by the finitely generated abelian group

$$H_{\mathbb{Z}} = 2\pi i\mathbb{Z} \subset \mathbb{C}, \qquad H_{\mathbb{C}} = H^{-1,-1},$$

is bigraded of type $(-1,-1)$, of rank 1. The m-tensor product $\mathbb{Z}(1) \otimes \cdots \otimes \mathbb{Z}(1)$ is a HS of weight $-2m$ denoted by $\mathbb{Z}(m)$:

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Remark: $2\pi i$ is an example of a **period**, a number written as a integral over a topical chain of an differential form (algebraic in some sense).

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4. the systems satisfy $W^\mathbb{C} := W \otimes \mathbb{C}$, such that the systems $\mathrm{Gr}_n^W H := (\mathrm{Gr}_n^W(H_{A\otimes\mathbb{Q}}), (\mathrm{Gr}_n^W(H_{A\otimes\mathbb{Q}}) \otimes \mathbb{C}, \mathrm{Gr}_n^{W^\mathbb{C}} H_\mathbb{C}, F))$ $\cong (\mathrm{Gr}_n^{W^\mathbb{C}} H_\mathbb{C}, (\mathrm{Gr}_n^{W^\mathbb{C}} H_\mathbb{C}, F))$are $A \otimes \mathbb{Q}$-HS of weight n

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$H_{\mathbb{Q}}$ is a MTHS if

$$\operatorname{Gr}_n^W H_{\mathbb{Q}} = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ \bigoplus \mathbb{Q}(-\frac{n}{2}), & \text{if } n \text{ is even.} \end{cases}$$

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Hence the main difference between MHS and MTHS is that of the weight filtration, i.e., the weight filtration of MTHS only is nontrivial only at even integers:

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Remark

Though the definition of MTHS is not widely used, it is MTHS which have connections with polylogrithms, which will appear in future lectures.

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Example of a variation of Hodge structure

Find your own!

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4. the filtrations $\mathcal{W}$ and $\mathcal{F}$ define an MHS on each fiber $(\mathcal{L}_{\mathcal{O}_X}(t), \mathcal{W}(t), \mathcal{L}(t))$ of the bundle $\mathcal{L}_{\mathcal{O}_X}$ at a point t.

Future topics

Future lectures plans

In the upcoming lectures we shall

Questions? With $\mathrm{Li}_n(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^n}$, stare at

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -\mathrm{Li}_1(z) & 2\pi i & 0 & \cdots & 0 \\ -\mathrm{Li}_2(z) & 2\pi i \log(z) & (2\pi i)^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\mathrm{Li}_n(z) & \frac{2\pi i \log(z)^{n-1}}{(n-1)!} & \frac{(2\pi i)^2 \log(z)^{n-2}}{(n-1)!} & \cdots & (2\pi i)^n \end{pmatrix}.$$

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- Play with multiple zeta values.

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