Basics of Hodge-de Rham Theory II: Hodge-de Rham Spectral Sequences and Periods of Mixed Hodge-Tate Structures

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Recap and Introduction

Recap and more motivations

Recall we introduced Hodge de-Rham theory as an extension of de Rham theory for (smooth) manifolds, today we shall see more of the Hodge theory aspects as we will introduce the Hodge-de Rham spectral sequences so we can calculation certain cohomology classes, and study mixed Hodge-Tate structures in detail, as it contains and relates to many arithmetic information and gadgets, such as multiple zeta values.

Today's Plan

We shall begin with a rough review of spectral sequences, then define and exploit its Hodge-de Rham, Frölicher, and hypercohomology versions to define the algebraic de Rham cohomology. Next we turn to examine one of the mixed Hodge structures, the mixed Hodge-Tate structures, its periods, and its relations to arithmetic objects.

(Review of) Spectral Sequences (1/4)

Jean Leray (1906-1988) was a prisoner of war from 1940 until 1945 and he was an expert in hydrodynamics. Not wishing the Germans to know his (real) expertise as he feared he would be forced to undertake war to work for them if they found out, he claimed to be a **topologist**, and worked only on topological problems for his time being a PoW. During those years, he invented **sheaves**, **sheaf cohomology** and **spectral sequences** for computing his sheaf cohomology: he and some of his fellow captives also organized a "université en captivité", and he became its rector. Spectral sequences were made algebraic by Koszul in 1945.

Differential Bigraded Module

A **differential bigraded module** over a ring R, is a indexed collection of R-modules, $\{E^{p,q}\}$, with $(p, q) \in \mathbb{Z} \times \mathbb{Z}$, together with an R-linear mapping, $d : E^{*,*} \to E^{*,*}$, the **differential** of bidegree $(s, 1 - s)$ or $(-s, s - 1)$, for some integer s, and satisfying $d \circ d = 0$.

(Review of) Spectral Sequences (2/4)

With the differential, we can take the **homology** of a bigraded module:

$$H^{p,q}(E^{*,*}, d) = \frac{\ker d : E^{p,q} \to E^{p+s,q-s+1}}{\operatorname{im} d : E^{p-s,q+s-1} \to E^{p,q}}$$

Spectral Sequence (SS)

An E_k-**spectral sequence** is a collection of differential bigraded R-modules $\{E_r^{*,*}, d_r\}$, where for $r \geq k$; the differentials are all of either

- bidegree $(-r, r-1)$ (for a spectral sequence of **homological type**)
- bidegree $(r, 1-r)$ (for a spectral sequence of **cohomological type**)

for all p, q, r, $E_{r+1}^{p,q}$ is isomorphic to $H^{p,q}(E_r^{*,*}, d_r)$.

Cycles and Boundaries of a SS

Given an E_k spectral sequence $\{E_r^{*,*}, d_r\}$, we define

$$Z_r := \{Z_r^{p,q} = \ker[d_r : E_r^{p,q} \to E_r^{p-r,q+r-1}]\}$$
$$B_r := \{B_r^{p,q} = \operatorname{im}[d_r : E_r^{p+r,q-r+q} \to E_r^{p,q}]\}$$

(Review of) Spectral Sequences (3/4)

Remark: Note that

$$B_r \subset Z_r \qquad \text{and} \qquad E_{r+1} = Z_r / B_r,$$

and applying induction for all $r \geq k$, we obtain

$$B_k \subset B_{k+1} \subset \cdots \subset B_r \subset \cdots \subset Z_r \subset \cdots \subset Z_{k+1} \subset Z_k.$$

Limit of the cycle, the boundary, and the spectral sequence

$$Z_\infty := \bigcap_{r \geq k} Z_r, \quad B_\infty := \bigcup_{r \geq k} B_r, \quad E_\infty := Z_\infty / B_\infty.$$

Degeneracy/Collapse and convergence/ abument

- $\{E_r^{*,*}, d_r\}$ is said to **degenerate** at the Nth sheet if $d_r = 0$ for $r \geq N$.
- E_r is said to **converge to**/abut to E_∞ if there is an $r(p,q) \geq k$ such that for all $r \geq r(p,q)$, the differential $d_r : E_r^{p,q} \to E_r^{p-r,q+r-1}$ is zero. This forces E_r to be isomorphic to E_∞ for r large, written
$$E_r^{p,q} \Rightarrow E_\infty^{p,q}.$$

(Review of) Spectral Sequences (4/4)

Confession

I lied a bit in the last slide, as we (really) require a **filtrated complex** to define the convergence of a spectral sequence.

Filtered Complex and Convergence of Spectral Sequences

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- Let H^n be a collection of objects with **finite** filtration

$$0 \subset FH^n \subseteq \cdots \subseteq F^t H^n = H^n$$

We say that $E_r^{p,q}$ converges to $H^\bullet$, and write $E_r^{p,q} \Rightarrow H^{p+q}$, if

$$\mathrm{gr}^p H^{p+1} := F^p H^{p+q} / F^{p+1} H^{p+q} = E_\infty^{p,q}$$

The Hodge-de Rham Spectral Sequence

Let M be a complex (smooth) manifold, $H_{dR}^{p+q}(M, \mathbb{C})$ its de Rham (Dolbeault) cohomology with complex coefficients.

The Spectral Sequence

$$E_1^{p,q} \Rightarrow \check{H}^p(M, \Omega^q_{M/k}) \cong H_{dR}^{p+q}(M, \mathbb{C}),$$

where $\Omega^q_{M/k}$ is the sheaf of differential q-forms on M over k. together with the usual spectral sequence resulting from a filtered object, in this case the Hodge filtration

$$F^i$$

The Frölicher Spectral Sequence

Let X be a complex manifold, and let (??·X,?) be its holomorphic de Rham complex. This complex is equipped with the ?naive? filtration (cf. section 8.3.1) Fp??· =???p. XX We have the corresponding filtration on the de Rham complex of X:

Hypercohomology Spectral Sequence

As in case of any hypercohomology, we have a spectral sequence for the de Rham cohomology:
There are two hypercohomology spectral sequences

- one with E_2-sheet
$$E_2^{p,q} \Rightarrow \mathbb{H}^p(R^q F(C))$$
- the other with E_1-sheet
$$R^p F(C^q)$$
and E_2-sheet
$$R^p F(H^q(C))$$
both converging to the hypercohomology
$$\mathbb{H}^{p+q}(C),$$
where $R^j\mathcal{F}$ is a right derived functor of $\mathcal{F}$.

Algebraic de Rham Cohomology

For a variety X over a field k, the second spectral sequence from above gives the Hodge-Rham spectral sequence for algebraic de Rham cohomology:

$$E_1^{p,q} = H^q(X, \Omega_X^p) \Rightarrow \mathbb{H}^{p+q}(X, \Omega_X^\bullet) =: H_{DR}^{p+q}(X/k).$$

Mixed Hodge Structure

Mixed Hodge Structure (MHS)

Let $A = \mathbb{Z}, \mathbb{Q}$, or $\mathbb{R}$. An A-mixed Hodge structure H consists of

1. an A-module of finite type H_A;
2. a finite increasing filtration $W_\bullet$ of $A \otimes \mathbb{Q}$-vector space $H_{A\otimes\mathbb{Q}}$ called the **weight filtration**:
$$\cdots \subseteq W_{m-1}H_{A\otimes\mathbb{Q}} \subseteq W_m H_{A\otimes\mathbb{Q}} \subseteq W_{m+1}H_{A\otimes\mathbb{Q}} \subseteq \cdots$$
3. a finite decreasing filtration $F^\bullet$ of the $\mathbb{C}$-vector space $H_\mathbb{C} = H_A \otimes_A \mathbb{C}$, called the Hodge filtration:
$$\cdots \supseteq F^{p-1}H_\mathbb{C} \supseteq F^p H_\mathbb{C} \supseteq F^{p+1}H_\mathbb{C} \supseteq \cdots$$
4. the systems satisfy $W^\mathbb{C} := W \otimes \mathbb{C}$, such that the systems $\mathrm{gr}_n^W H := (\mathrm{gr}_n^W(H_{A\otimes\mathbb{Q}}), (\mathrm{gr}_n^W(H_{A\otimes\mathbb{Q}}) \otimes \mathbb{C}, \mathrm{Gr}_n^{W^\mathbb{C}} H_\mathbb{C}, F))$ $\cong (\mathrm{Gr}_n^{W^\mathbb{C}} H_\mathbb{C}, (\mathrm{Gr}_n^{W^\mathbb{C}} H_\mathbb{C}, F))$are $A \otimes \mathbb{Q}$-HS of weight n

Mixed Hodge Structures (MHS) (1/2)

P. Deligne was responsible to developing mixed Hodge Structures, and A. Beilinson extended it to mixed motives

Mixed Hodge-Tate Structures (1/2)

Mixed Hodge-Tate Structures (2/2)

Mixed

A morphism ? : A ? B between mixed Hodge-Tate structures consists of a homomorphism ? : A? ? B? such that the induced homomorphisms AQ ? BQ and AC ? BC preserve the weight and Hodge filtrations, respectively.

Hodge decomposition theorem

Periods of Mixed Hodge-Tate Structures (1/2)

There can be many mixed Hodge-Tate structures with the same weight graded quotients. These are parameterized by a moduli space whose coordinates are the periods. The basic idea is quite simple. Suppose that V is a ?-mixed HodgeTate structure. For simplicity, we suppose that V? is torsion free and that GrW ?2m V? ?= ?(m) rm .

Periods of Mixed Hodge-Tate Structures (1/2)

Polylogrithms

Multiple Zeta Values (1/2)

Multiple Zeta Values (2/2)

Recap and Future topics

Recap of the past two lectures

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Questions?

Future lectures plans

- Scrutinize Chen's de Rham Theory of homotopy groups defined using iterated integrals.
- Relate iterated integrals to multiple zeta values.
- Develop the Hodge-de Rham theory of the unipotent fundamental group of $\mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}$ using the above techniques.