Basics of Hodge-de Rham Theory III: Iterated Integrals and Chen's π_1 de Rham Theorem

Justin Scarfy

The University of British Columbia

2015-02-04

Motivational Problem

Let M be a smooth manifold and $\alpha, \beta \in P_{a,a}(M)$. Then for any (closed or not) 1-form ω on M,

$$\int_{\alpha\beta} \omega = \int_{\alpha} \omega + \int_{\beta} \omega = \int_{\beta} \omega + \int_{\alpha} \omega = \int_{\beta\alpha} \omega,$$

i.e., ordinary line integrals are intrinsically abelian – they are unable to detect the order in which we compose α and β. Hence, ordinary line integrals cannot detect elements of the commutator subgroup of $\pi_1(M, a)$.

Motivational Problem

Let M be a smooth manifold and $\alpha, \beta \in P_{a,a}(M)$. Then for any (closed or not) 1-form ω on M,

$$\int_{\alpha\beta} \omega = \int_{\alpha} \omega + \int_{\beta} \omega = \int_{\beta} \omega + \int_{\alpha} \omega = \int_{\beta\alpha} \omega,$$

i.e., ordinary line integrals are intrinsically abelian – they are unable to detect the order in which we compose α and β. Hence, ordinary line integrals cannot detect elements of the commutator subgroup of $\pi_1(M, a)$.

Fundamental Question

How can we use differential forms to detect elements of $\pi_1(M, a)$ that vanish in $H_1(M, \mathbb{R})$?

Motivational Problem

Let M be a smooth manifold and $\alpha, \beta \in P_{a,a}(M)$. Then for any (closed or not) 1-form ω on M,

$$\int_{\alpha\beta} \omega = \int_{\alpha} \omega + \int_{\beta} \omega = \int_{\beta} \omega + \int_{\alpha} \omega = \int_{\beta\alpha} \omega,$$

i.e., ordinary line integrals are intrinsically abelian – they are unable to detect the order in which we compose α and β. Hence, ordinary line integrals cannot detect elements of the commutator subgroup of $\pi_1(M, a)$.

Fundamental Question

How can we use differential forms to detect elements of $\pi_1(M, a)$ that vanish in $H_1(M, \mathbb{R})$?

Kuo-Tsai Chen (1923-1987)'s Answer

Chen gave a non-abelian generalization of the standard line integral: these are called iterated line integrals

Definitions (1/2)

Time Ordered Integral

Let $\omega_1, \ldots, \omega_r$ be smooth 1-forms on M with values in an associated $\mathbb{R}$ algebra $A = \mathbb{C}, \mathsf{M}_n(\mathbb{R})$ or $\mathsf{M}_n(\mathbb{C})$, i.e., $\omega_j \in \Omega^1_{\mathbb{R}}(M) \otimes_{\mathbb{R}} A$. Suppose $\gamma \in PM$, define

$$\int_\gamma \omega_1 \cdots \omega_r \in A := \int_{0 \le t_1 \le t_2 \le \cdots \le t_r \le 1} f_1(t_1) \cdots f_r(t_r)\, dt_1 \cdots dt_r,$$

where $\gamma^* \omega_j = f_j(t)\, dt$, called the time ordered integral. The integral is to be viewed as function

$$\int \omega_1 \cdots \omega_r : PM \to A.$$

Definitions (1/2)

Time Ordered Integral

Let $\omega_1, \ldots, \omega_r$ be smooth 1-forms on M with values in an associated $\mathbb{R}$ algebra $A = \mathbb{C}, \mathsf{M}_n(\mathbb{R})$ or $\mathsf{M}_n(\mathbb{C})$, i.e., $\omega_j \in \Omega^1_{\mathbb{R}}(M) \otimes_{\mathbb{R}} A$. Suppose $\gamma \in PM$, define

$$\int_\gamma \omega_1 \cdots \omega_r \in A := \int_{0 \le t_1 \le t_2 \le \cdots \le t_r \le 1} f_1(t_1) \cdots f_r(t_r)\, dt_1 \cdots dt_r,$$

where $\gamma^* \omega_j = f_j(t)\, dt$, called the time ordered integral. The integral is to be viewed as function

$$\int \omega_1 \cdots \omega_r : PM \to A.$$

Homotopy Functional

With S a set, a function $\mathcal{F} : PM \to S$ is called a homotopy functional if the value of $\mathcal{F}$ on a path γ only depends on its homotopy class in $P_{a,b}M$.

Definitions (2/2)

Homotopy Functional explained

More precisely, for each pair of points $a, b \in M$, there is a function $f_{a,b} : \pi(M; a, b) \to S$ such that the diagram below commutes

$$
\begin{tikzcd}
P_{a,b}M \arrow[r, "\mathcal{F}"] \arrow[d] & S \\
\pi(M; a, b) \arrow[ur, "f_{a,b}"'] &
\end{tikzcd}
$$

Definitions (2/2)

Homotopy Functional explained

More precisely, for each pair of points $a, b \in M$, there is a function $f_{a,b} : \pi(M; a, b) \to S$ such that the diagram below commutes

$$\begin{tikzcd} P_{a,b}M \arrow[r, "\mathcal{F}"] \arrow[d] & S \\ \pi(M; a, b) \arrow[ur, "f_{a,b}"'] & \end{tikzcd}$$

A homotopy functional $\mathcal{F} : PM \to S$ induces a function $\phi_{\mathcal{F}} : \pi_1(M, a) \to S$ by taking the homotopy class of a loop γ to $\mathcal{F}(\gamma)$. More generally, it induces functions $\phi_{\mathcal{F}} : \pi(M; a, b) \to S$.

Definitions (2/2)

Homotopy Functional explained

More precisely, for each pair of points $a, b \in M$, there is a function $f_{a,b} : \pi(M; a, b) \to S$ such that the diagram below commutes

$$\begin{tikzcd}
P_{a,b}M \arrow[r, "\mathcal{F}"] \arrow[d] & S \\
\pi(M; a, b) \arrow[ur, "f_{a,b}"'] &
\end{tikzcd}$$

A homotopy functional $\mathcal{F} : PM \to S$ induces a function $\phi_{\mathcal{F}} : \pi_1(M, a) \to S$ by taking the homotopy class of a loop γ to $\mathcal{F}(\gamma)$. More generally, it induces functions $\phi_{\mathcal{F}} : \pi(M; a, b) \to S$.

Restatement of the Question we are after

The basic problem, then, is to find all iterated integrals that are homotopy functionals.

Basic Properties of Iterated Integrals (1/6)

Let $f : M \to N$ is a smooth mapping between smooth manifolds, and $\omega_1, \ldots, \omega_r \in \Omega^1(N)$ and $\alpha \in PM$, then

$$\int_{f \circ \alpha} \omega_1 \cdots \omega_r = \int_\alpha f^*\omega_1 f^*\omega_2 \cdots f^*\omega_r. \quad \square$$

Basic Properties of Iterated Integrals (1/6)

Let $f : M \to N$ is a smooth mapping between smooth manifolds, and $\omega_1, \ldots, \omega_r \in \Omega^1(N)$ and $\alpha \in PM$, then

$$\int_{f \circ \alpha} \omega_1 \cdots \omega_r = \int_\alpha f^*\omega_1 f^*\omega_2 \cdots f^*\omega_r. \quad \square$$

Iterated integrals also enjoy a combinatorial nature

They reflect the combinatorics of simplices, and we exploit this property to

Basic Properties of Iterated Integrals (1/6)

Let $f : M \to N$ is a smooth mapping between smooth manifolds, and $\omega_1, \ldots, \omega_r \in \Omega^1(N)$ and $\alpha \in PM$, then

$$\int_{f\circ\alpha} \omega_1 \cdots \omega_r = \int_\alpha f^*\omega_1 f^*\omega_2 \cdots f^*\omega_r. \quad \Box$$

Iterated integrals also enjoy a combinatorial nature

They reflect the combinatorics of simplices, and we exploit this property to

1. evaluate an iterated integral on the product of two paths,

Basic Properties of Iterated Integrals (1/6)

Let $f : M \to N$ is a smooth mapping between smooth manifolds, and $\omega_1, \ldots, \omega_r \in \Omega^1(N)$ and $\alpha \in PM$, then

$$\int_{f\circ\alpha} \omega_1 \cdots \omega_r = \int_\alpha f^*\omega_1 f^*\omega_2 \cdots f^*\omega_r. \quad \square$$

Iterated integrals also enjoy a combinatorial nature

They reflect the combinatorics of simplices, and we exploit this property to

1. evaluate an iterated integral on the product of two paths,
2. pointwise multiply two iterated integrals (as functions on PM).

Basic Properties of Iterated Integrals (1/6)

Let $f : M \to N$ is a smooth mapping between smooth manifolds, and $\omega_1, \ldots, \omega_r \in \Omega^1(N)$ and $\alpha \in PM$, then

$$\int_{f\circ\alpha} \omega_1 \cdots \omega_r = \int_\alpha f^*\omega_1 f^*\omega_2 \cdots f^*\omega_r. \quad \square$$

Iterated integrals also enjoy a combinatorial nature

They reflect the combinatorics of simplices, and we exploit this property to

1. evaluate an iterated integral on the product of two paths,
2. pointwise multiply two iterated integrals (as functions on PM).
3. evaluate an iterated integral on the inverse of a path.

Basic Properties of Iterated Integrals (1/6)

Let $f : M \to N$ is a smooth mapping between smooth manifolds, and $\omega_1, \ldots, \omega_r \in \Omega^1(N)$ and $\alpha \in PM$, then

$$\int_{f\circ\alpha} \omega_1 \cdots \omega_r = \int_\alpha f^*\omega_1 f^*\omega_2 \cdots f^*\omega_r. \quad \square$$

Iterated integrals also enjoy a combinatorial nature

They reflect the combinatorics of simplices, and we exploit this property to

1. evaluate an iterated integral on the product of two paths,
2. pointwise multiply two iterated integrals (as functions on PM).
3. evaluate an iterated integral on the inverse of a path.

Our model for the standard r-simplex is the time ordered r-simplex:

$$\Delta^r = \{(t_1, \ldots, t_r) \in R^r : 0 \le t_1 \le \ldots \le t_r \le 1\}.$$

Basic Properties of Iterated Integrals (2/6)

Now the definition of a basic iterated line integral may be viewed as:

$$\int_\gamma \omega_1 \cdots \omega_r = \int_{\Delta^r} (p_1^* \gamma^* \omega_1) \wedge \cdots \wedge (p_r^* \gamma^* \omega_r),$$

with $p_j : \mathbb{R}^r \to \mathbb{R}$ being the projection onto the j^{th} coordinate.

Basic Properties of Iterated Integrals (2/6)

Now the definition of a basic iterated line integral may be viewed as:

$$\int_\gamma \omega_1 \cdots \omega_r = \int_{\Delta^r} (p_1^*\gamma^*\omega_1) \wedge \cdots \wedge (p_r^*\gamma^*\omega_r),$$

with $p_j : \mathbb{R}^r \to \mathbb{R}$ being the projection onto the j^{th} coordinate.

Facts about time ordered r-simplex

With $t_0 = 0$ and $t_{r+1} = 1$,

$$\Delta^r = \bigcup_{j=0}^{r} \{(t_1, \ldots, t_r) : 0 \le t_1 \le \cdots \le t_j \le 1/2 \le t_{j+1} \le \cdots \le t_r\}$$

and that there is a natural identification of $\Delta^j \times \Delta^{r-j}$ with

$$\{(t_1, \ldots, t_r) : 0 \le t_1 \le \cdots \le t_j \le 1/2 \le t_{j+1} \le \cdots \le t_r\}.$$

Basic Properties of Iterated Integrals (3/6)

More facts about time ordered r-simplex

Viewing $\Delta^r \times \Delta^s \subset \mathbb{R}^r \times \mathbb{R}^s = \mathbb{R}^{r+s}$, we have

$$\Delta^s \times \Delta^s = \bigcup_{\sigma \in \mathrm{Sh}(r,s)} : \{(t_1, \dots, t_{r+s}) : 0 \le t_{\sigma(1)} \le \cdots \le t_{\sigma(r+s)} \le 1\},$$

where $\mathrm{Sh}(r,s)$ is the set of shuffle of type (r,s) of permutations σ of $\{1, 2, \dots, r+s\}$, i.e., a permutation with

$$\sigma^{-1}(1) < \cdots < \sigma^{-1}(r) \quad \text{and} \quad \sigma^{-1}(r+1) < \cdots < \sigma^{-1}(r+s).$$

Basic Properties of Iterated Integrals (3/6)

More facts about time ordered r-simplex

Viewing $\Delta^r \times \Delta^s \subset \mathbb{R}^r \times \mathbb{R}^s = \mathbb{R}^{r+s}$, we have

$$\Delta^s \times \Delta^s = \bigcup_{\sigma \in \mathrm{Sh}(r,s)} : \{(t_1, \ldots, t_{r+s}) : 0 \leq t_{\sigma(1)} \leq \cdots \leq t_{\sigma(r+s)} \leq 1\},$$

where $\mathrm{Sh}(r,s)$ is the set of shuffle of type (r,s) of permutations σ of $\{1, 2, \ldots, r+s\}$, i.e., a permutation with

$$\sigma^{-1}(1) < \cdots < \sigma^{-1}(r) \quad \text{and} \quad \sigma^{-1}(r+1) < \cdots < \sigma^{-1}(r+s).$$

When $\omega_1, \omega_2, \ldots$ are smooth 1-forms on the manifold M.

- Coproduct: If $\alpha, \beta \in PM$ are composable (i.e. $\alpha(1) = \beta(0)$), then

$$\int_{\alpha\beta} \omega_1 \cdots \omega_r = \sum_{j=0}^{r} \int_\alpha \omega_1 \cdots \omega_j \int_\beta \omega_{j+1} \cdots \omega_r.$$

Basic Properties of Iterated Integrals (4/6)

- Shuffle Product: If $\alpha \in PM$, then

$$\int_\alpha \omega_1 \cdots \omega_r \int \omega_{r+1} \cdots \omega_{r+s} = \sum_{\sigma \in \mathrm{Sh}(r,s)} \omega_{\sigma(1)} \omega_{\sigma(2)} \cdots \omega_{\sigma(r+s)}.$$

- Antipode: If $\alpha \in PM$, then

$$\int_{\alpha^{-1}} \omega_1 \cdots \omega_r = (-1)^r \int_\alpha \omega_r \cdots \omega_1.$$

Iterated Integrals are Unique up to Path-Homotopy

Iterated integrals $\int \omega_1 \cdots \omega_r : PM \to A$ factor through the quotient mapping $PM \to PM/\sim$, i.e. if $\alpha \sim \beta \in PM$, then

$$\int_\alpha \omega_1 \cdots \omega_r = \int_\beta \omega_1 \cdots \omega_r.$$

□

Basic Properties of Iterated Integrals (5/6)

The set $(P_{x,x}M)/\sim$ has a well defined associative product
$$[(P_{x,x}M)/\sim] \times [(P_{x,x}M)/\sim] \to (P_{x,x}M)/\sim .$$

Denote the constant path at x by 1_x, and set

$$P(M,x) := \coprod_{(P_{x,x}M)/\sim} \mathbb{Z},$$

as an associative algebra whose elements are formal finite linear combinations

$$c := \sum_{\gamma \in P_{x,x}M} n_\gamma \gamma.$$

Iterated integrals with values in A thus define functions

$$\int \omega_1 \cdots \omega_r : P(M,x) \to A,$$

$$c \mapsto \left\langle \int \omega_1 \cdots \omega_r, c \right\rangle.$$

Basic Properties of Iterated Integrals (6/6)

Nilpotence of Iterated Integrals

Let $r, s \geq 1, \omega_1, \ldots, \omega_r \in \Omega^1(M)$ and $\alpha_1, \ldots, \alpha_s \in P(M, x)$, then

$$\left\langle \int \omega_1 \cdots \omega_r, (\alpha_1 - 1_x) \cdots (\alpha_s - 1_x) \right\rangle = \begin{cases} \prod_{j=1}^r \int_{\alpha_j} \omega_j & \text{if } s \leq r, \\ 0 & \text{if } s > r. \end{cases}$$

Remark and an Example

This generalizes the standard line integrals ($r = 1$) as

$$\begin{aligned} \left\langle \int \omega, (\alpha_1 - 1_x)(\alpha_2 - 1_x) \right\rangle &= \left\langle \int \omega, \alpha_1\alpha_2 - \alpha_1 - \alpha_2 + 1_x \right\rangle \\ &= \int_{\alpha_1\alpha_2} \omega - \int_{\alpha_1} \omega - \int_{\alpha_2} \omega + \int_{1_x} \omega = 0. \end{aligned}$$

A Word on its Proof

Chen used group algebra to

The Group Algebra and its Dual (1/5)

Group Algebra

Let G be a discrete group and R a commutative ring with 1. Denote the **group algebra** of G over R by $R[G]$ with the definition

$$R[G] := \left\{ \sum_{g \in G}^{\text{finite}} r_g g : r_g \in R \right\}.$$

The **augmentation** is the homomorphism $\epsilon : R[G] \to R$ defined by

$$\epsilon : \sum_{g \in G} r_g g \mapsto \sum_{g \in G} r_g.$$

The kernel of ϵ is called the **augmentation ideal** and denoted J_R, and powers of J_R,

$$R[G] = J_R^0 \supseteq J_R \supseteq J_R^2 \supset \cdots$$

define a topology, called the J_R-**adic topology**, on $R[G]$.

The Group Algebra and its Dual (2/5)

J_R-adic Completion of $R[G]$

Note that this topology is not usually separated, i.e., the intersection of the powers of J_R is not always trivial. The J-adic completion of $R[G]$ is

$$\widehat{R[G]} := \varprojlim_m R[G]/J^m.$$

Facts

The graded algebra

$$\bigoplus_{m=0}^{\infty} J_R^m / J_m^{R+1}$$

is generated by J_R/J_R^2, and a section of the projection $\widehat{J_R} \to J_R/J_R^2$ induces an algebra homomorphism

$$T(J_R/J_R^2) \to \widehat{R[G]},$$

The Group Algebra and its Dual (3/5)

Facts

with dense image, where

$$T(V) := R \oplus \bigoplus_{m>0} V^{\oplus m}$$

denotes the free associative R-algebra generated by the R-module V .
Hence if $H_1(\pi; R)$ is a free R-module, then $\widehat{R[G]}$ is the quotient of the completed tensor algebra

$$T(\widehat{H_1(G;R)})$$

generated by $H_1(G; R)$ where the projection $T(\widehat{H_1(G;R)}) \to \widehat{R[G]}$ induces the identity

$$H_1(\pi; R) \cong I/I^2 \to J/J^2 \cong H_1(G; R).$$

The Group Algebra and its Dual (4/5)

Continuous Group Algebra Homomorphisms

For a discrete R-module N, define

$$\mathrm{Hom}_R^{\mathsf{cts}}(R[G], N) := \varinjlim_m \mathrm{Hom}_R(R[G]/J^m, N),$$

and its continuous dual

$$\mathrm{Hom}_R^{\mathsf{cts}}(R[G], R) = \mathrm{Hom}_R^{\mathsf{cts}}(\mathbb{Z}[G], R)$$

is a commutative R-algebra whose product is pointwise multiplication of functions.

Hopf Algebra

An augmented bialgebra is an K-algebra (K a field) $H \to K$ with a homomorphism, $\Delta : H \to H \times H$, called the **comultiplication**. A **commutative Hopf algebra** is an augmented bialgebra together with a homomorphism $S : A \to A$, called the antipode, which is compatible with

The Group Algebra and its Dual (5/5)

the augmentation, multiplication and comultiplication, i.e., see diagram on the board.

The Dual Filtration of the J-adic topology

$$R = B_0\mathrm{Hom}_R^{\mathsf{cts}}(R[G], R) \subseteq B_1\mathrm{Hom}_R^{\mathsf{cts}}(R[G], R) \subseteq B_2\mathrm{Hom}_R^{\mathsf{cts}}(R[G], R) \subseteq \cdots$$

of $\mathrm{Hom}_R^{\mathsf{cts}}(R[G], R)$, where

$$B_m\mathrm{Hom}_R^{\mathsf{cts}}(R[G], R) := \mathrm{Hom}_R^{\mathsf{cts}}(R[G]/J^{m+1}, R),$$

hence, $\mathrm{Hom}_R^{\mathsf{cts}}(R[G], R)$ is a filtered Hopf algebra, i.e. is the multiplication, comultiplication and antipode induce mappings

$$B \otimes B_m \to B_{m+n}, \text{by } B_n \mapsto \sum_{j+k=n} B_j \otimes B_k \text{ and } B_m \mapsto B_m.$$

Chen's dR Theorem for the Fundamental Group (1/3)

$\mathrm{Ch}(P_{x,y}M; F)$

Let M be a connected manifold, $x, y, z \in M$ and that $F = \mathbb{R}$ or $\mathbb{C}$. Denote the set of iterated integrals $PM \to F$ restricted to $P_{x,y}M$ by $\mathrm{Ch}(P_{x,y}M; F)$.

- The shuffle product formula implies that this is an F-algebra.
- The coproduct formula implies that the mapping

$$\mathrm{Ch}(P_{x,z}M; F) \to \mathrm{Ch}(P_{x,y}M; F) \otimes_F \mathrm{Ch}(P_{y,z}M : F)$$

$$\int \omega_1 \cdots \omega_r \mapsto \sum_{j=0}^{r} \int \omega_1 \cdots \omega_j \otimes \int \omega_{j+1} \cdots \omega_r$$

is well defined and is dual to path multiplication

$$P_{x,y}M \times P_{y,x}M \to P_{x,x}M.$$

Chen's dR Theorem for the Fundamental Group (2/3)

Remarks on $\mathrm{Ch}(P_{x,y}M; F)$

- When $x = y$, this is augmented by evaluation at the constant loop 1_x.
- With this augmentation, shuffle product, and coproduct, $\mathrm{Ch}(P_{x,x}M; F)$ is a **commutative Hopf algebra**.

Length of Iterated Integrals and its Filtration

Iterated integrals are naturally filtered by **length**, hence we can denote the linear span of the $\int \omega_1 \cdots \omega_r$, $r \leq n$ by $L_n\mathrm{Ch}(P_{x,y}M; F)$. With these filtrations, $\mathrm{Ch}(P_{x,y}M; F)$ is a **filtered Hopf algebra**.

Remarks on the filtration

It may appear that iterated integrals are graded by length. However, as

$$\int \omega_1 \cdots \omega_{j-1}(df)(\omega_j) \cdots \omega_r = \int \omega_1 \cdots \omega_{j-1} f(\omega_j)\omega_{j+1} \cdots \omega_r - \int \omega_1 \cdots \omega_{j-1}(f\omega_{j-1})\omega_j \cdots \omega_r,$$

iterated integrals are only filtered by length.

Chen's dR Theorem for the Fundamental Group (3/4)

Subspace of iterated integrals that are homotopy functionals

Let $H^0(\mathrm{Ch}(P_{x,y}M;F))$ denote the subspace consisting of those iterated integrals that are homotopy functionals. It is clearly a subring of $\mathrm{Ch}(P_{x,y}M;F)$ as the product of two homotopy functionals is a homotopy functional. The length filtration restricts to a length filtration $L_\bullet$ of $H^0(\mathrm{Ch}(P_{x,y}M;F))$.

Properties of $H^0(\mathrm{Ch}(P_{x,y}M;F)$

The coproduct and antipode restrict to a coproduct

$$H^0(\mathrm{Ch}(P_{x,z}M;F)) \to H^0(\mathrm{Ch}(P_{x,y}M;F)) \otimes_F H^0(\mathrm{Ch}(P_{y,z}M;F)),$$

and the antipode

$$H^0(\mathrm{Ch}(P_{x,z}M;F)) \to H^0(\mathrm{Ch}(P_{x,y}M;F)).$$

give us that $H^0(\mathrm{Ch}(P_{x,x}M;F))$ is a **filtered commutative Hopf**

Chen's dR Theorem for the Fundamental Group (4/4)

Integration on $H^0(\mathrm{Ch}(P_{x,y}(M;F))$

Integration induces a injective mapping

$$\int : H^0(\mathrm{Ch}(P_{x,y}(M;F)) \to \mathrm{Hom}_F^{\mathsf{cts}}(\mathbb{Z}\pi_1(M,x),F),$$

since the set of path components of $P_{x,x}M$ is $\pi_1(M,x)$, and as $H^0(\mathrm{Ch}(P_{x,y}M;F))$ is, by definition, a subset of functions on PM.

One version of Chen's de Rham Theorem for the fundamental groups

The above homomorphism is surjective, and therefore an isomorphism of Hopf algebras. Moreover, it is an isomorphism of filtered Hopf algebras. That is, for each $m \geq 0$, integration induces an isomorphism

$$L_m H^0(\mathrm{Ch}(P_{x,y}M;F)) \cong \mathrm{Hom}_F^{\mathsf{cts}}(\mathbb{Z}\pi_1(M,x)/J^{m+1},F).$$

Proof of Chen's Theorem in a Special Case

Remark

If the manifold $M = \mathbb{P}^1(\mathbb{C})$, then as it is simply connected, there is nothing to prove. Hence we remove

The holomorphic 1-forms on U with logarithmic poles on S, $H^0(\Omega^1_{\mathbb{P}^1}(\log S))$ has basis

$$\omega_j := \frac{dz}{z - a_j}, \quad j = 1, \ldots, N.$$

Denote the set of iterated integrals built up from elements of $H^0(\Omega^1_{\mathbb{P}^1}(\log S))$ by $\mathrm{Ch}(H^0(\Omega^1_{\mathbb{P}^1}(\log S)))$.