Basics of Hodge-de Rham Theory IV: Hodge Theory for the Fundamental Group(oid) and Drinfeld Associator

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Proof of Chen's dR Theorem in a Special Case (1/4)

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is surjective, and therefore an isomorphism of Hopf algebras. Moreover, it is an isomorphism of filtered Hopf algebras, i.e., for each $m \geq 0$, integration induces an isomorphism

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and promised to give a proof to the special case when M is a Zariski open subset of $\mathbb{P}^1(\mathbb{C})$, i.e., $M = \mathbb{P}^1(\mathbb{C}) \setminus S$, S a finite subset of $\mathbb{P}^1(\mathbb{C})$.

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Denote $\mathrm{Ch}(H^0(\Omega^1_{\mathbb{P}^1}(\log S)))$ the set of iterated integrals built up from elements of $H^0(\Omega^1_{\mathbb{P}^1}(\log S))$.

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Proposition

For each $x \in U$, the composite

$$\mathrm{Ch}(H^0(\Omega^1_{\mathbb{P}^1}(\log S))) \hookrightarrow H^0(\mathrm{Ch}(P_{x,x}U;\mathbb{C})) \hookrightarrow \mathrm{Hom}^{\mathsf{cts}}_{\mathbb{Z}}(\mathbb{Z}\pi_1(U,x),\mathbb{C})$$

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Sketch of Proof.
Define $\epsilon : A := \mathbb{C}[[X_1,\ldots,X_N]] \to \mathbb{C}$, the augmentation map by taking a power series to its constant term, and hence the augmentation ideal $\ker\epsilon$ is the maximal ideal $(X_1,\ldots,X_N)$.

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Now view the formal power series

$$T = 1+\sum_j \int \omega_j X_j + \sum_{j,k} \int \omega_j\omega_k X_j X_K + \cdots \in \mathrm{Ch}(H^0(\Omega^1_{\mathbb{P}^1}(\log S)))[[X_1,\ldots$$

as an A-valued iterated integral, where the coefficients of the monomial $X_{i_1}\ldots X_{i_r}$ is $\int \omega_{i_1}\cdots\omega_{i_r}$.

Proof of Chen's dR Theorem in a Special Case (4/4)

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Claim

The mapping $\widehat{\Theta}$ is an isomorphism.

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T(\widehat{H_1(U;\mathbb{C}}) \arrow[d, "\Phi"'] \arrow[dr, "\widehat{\Theta}\circ\Phi"] & \\
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Now you may verify that $\widehat{\Theta} \circ \Phi$ induces an isomorphism on I/I^2, and thus $\widehat{\Theta}$ is an isomorphism. This in turn implies that in

$$\mathrm{Ch}(H^0(\Omega^1_{\mathbb{P}^1}(\log S))) \hookrightarrow H^0(\mathrm{Ch}(P_{x,x}U;\mathbb{C})) \hookrightarrow \mathrm{Hom}^{\mathsf{cts}}_{\mathbb{Z}}(\mathbb{Z}[\pi_1(U,x)],\mathbb{C}),$$

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Remark

You can play the same game on fundamental groupoids for smooth manifolds.

Hodge Theory for the Fundamental Group(oid) (1/4)

For $A = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$, recall a A-mixed Hodge structure (MHS) H consisted of an A-module of finite type, with two finite filtrations, an increasing weight filtration $W_\bullet$ on the rational vector space, and a decreasing Hodge filtration $F^\bullet$ on the complexified vector space that satisfy some intricate complementary triple filtrations conditions.

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A natural MHTS

We now construct a natural mixed Hodge-Tate structure on

$$V_{\mathbb{Z}} := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\pi_1(U,x)]/J^{n+1}, \mathbb{Z}),$$

where its complexification,

$$V_{\mathbb{C}} := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\pi_1(U,x)]/J^{n+1}, \mathbb{C}),$$

is identified with $L_n \operatorname{Ch}(H^0(\Omega^1_{\mathbb{P}^1}(\log S))$ by the isomorphism we just proved.

Hodge Theory for the Fundamental Group(oid) (2/4)

- Define the weight filtration by

$$W_{2m}V_{\mathbb{Q}} := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\pi_1(U,x)]/J^{n+1}, \mathbb{Q}),$$

and note by Chen's dR Theorem, the complexified weight filtration is simply the length filtration:

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Hence these filtrations defined a MHS on V.

Hodge Theory for the Fundamental Group(oid) (3/4)

In addition, since the $2m^{\text{th}}$ weight graded quotient

$$\mathrm{Gr}^W_{2m} V := W_{2m}V/W_{2m-2}V = \bigoplus \mathbb{Q}(-m),$$

it is a Mixed Hodge-Tate structure (MHTS), and complexification the space of integrated integrals

$$\{\int \omega_{j_1} \cdots \omega_{j_m} : \omega_{j_k} \in H^0(\Omega^1_{\mathbb{P}^1}(\log S))\} \cong H^1(U;\mathbb{C})^{\otimes m},$$

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Hodge Theory for the Fundamental Groupoid

The same method can be used to define a mixed Hodge-Tate structure on

$$H_0(P_{x,y}(\mathbb{P}^1(\mathbb{C}) \setminus S))/J_{x,y}^{n+1},$$

a fundamental groupoid.

Preamble to Mixed Tate Motives (1/2)

Philosophy of motives over $\mathrm{Spec}\mathbb{Z}$

Motives over $\mathrm{Spec}\mathbb{Z}$ should arise as invariants (cohomology, homotopy, etc.) of varieties (and stacks) defined over $\mathbb{Z}$ that have good reduction at every prime number. Obvious examples include the projective spaces $\mathbb{P}^N_{\mathbb{Z}}$ and the moduli stacks of curves $\mathcal{M}_{g,n}$.

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For now we are interested in the open subsets of the projective line:

$$U_{\mathbb{Z}} = \mathbb{P}^1_{\mathbb{Z}} \setminus S := \mathrm{Spec}\mathbb{Z}[z, t_1, \ldots, t_N]/(z - a_j)t_j - 1 : j = 1, \ldots, N),$$

with $S = \{a_1, \ldots, a_N, \infty\}$, as usual, and each $a_j \in \mathbb{Z}$.
This has good reduction at the prime p if he cardinality of $S \pmod{p}$ equals that of S, Now take $U = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. In order to consider the fundamental group of U, we need a base point x. If we choose $x \in \mathbb{Z} \setminus \{0, 1\}$, then the pair (U, x) has bad reduction at the prime p whenever $p \mid x(x-1)$ as then the base point reduces to 0 or 1, which are not in $U(\mathbb{F}_p)$.

Preamble to Mixed Tate Motives (2/2)

Necessity of asymptotic base points

This forces us to consider "asymptotic base points." These are tangent vectors of $\mathbb{P}^1_{\mathbb{Z}}$ at $\{0, 1, \infty\}$ that are non-zero at each prime p, such as

$$\vec{01} := \partial/\partial z \in T_0\mathbb{P}^1 \quad \text{and} \quad \vec{10} := -\partial/\partial z \in T_1\mathbb{P}^1.$$

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Moreover, the **tannakian category** of $\mathbb{Q}$-mixed Tate motives over $\mathrm{Spec}\mathbb{Z}$ does exist via the works of Voevodsky, Levine and Goncharov. Deligne and Goncharov have shown that the direct system

$$0 \hookrightarrow \mathbb{Z} \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\pi_1(U,x)/J^2], \mathbb{Z}) \hookrightarrow \cdots \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\pi_1(U,x)/J^n], \mathbb{Z}) \hookrightarrow \cdots$$

of the $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}[\pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}], \vec{01})/J^{n+1}, \mathbb{Q})$ is a directed system in this category.

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of the $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}[\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}], \vec{01})/J^{n+1}, \mathbb{Q})$ is a directed system in this category. We shall exploit the topological and Hodge theoretic aspects of $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \vec{01})$ and $\pi_0(P_{\vec{01}, \vec{10}}(\mathbb{P}^1 \setminus \{0, 1, \infty\}))$ in the rest of this lecture.

Drinfeld Associator - a premier (1/2)

Now consider the fundamental groupoid $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ with objects the two tangent vectors $\vec{01} \in T_0\mathbb{P}^1$ and $\vec{10} \in T_1\mathbb{P}^1$. This is generated by the paths [see whiteboard], where $\sigma_0 \in P_{\vec{01},\vec{01}}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$ and $\sigma_1 \in P_{\vec{10},\vec{10}}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$

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Set

$$\Phi(X_0, X_1) := \widehat{\Theta}_{\vec{01},\vec{10}}([0, 1]) = \lim_{t \to 0} t^{X_0} T([t, 1-t]) t^{X_1} \in A.$$

This is known as the Drinfeld associator and was first constructed by V. Drinfeld.

Drinfeld Associator - a premier (2/2)

Theorem

The periods of the limit mixed Hodge-Tate structure on $\mathbb{Q}[\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{01})^\wedge]$ is precisely $\mathsf{MZV}_{\mathbb{C}} := \mathsf{MZV} \oplus i\pi\mathsf{MZV}$.

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Proof

Since $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \vec{01})$ is generated by the paths σ_0 and $[0, 1]\sigma_1[1, 0]$,

$$\widehat{\Theta}_{\vec{01},\vec{01}}(\sigma_0) = e^{2\pi i X_0} \quad \text{and} \quad \widehat{\Theta}_{\vec{01},\vec{01}}(\sigma_1) = e^{-2\pi i X_1}$$

and the fact that the coefficients of $\Phi(X_0, X_1)$ are MZVs. [Blackbox]

Summary and Future Directions

Recall in the past 4 lectures we learned

1. Basics of Hodge theory.
2. Hodge-de Rham Spectral Sequences.
3. Iterated Integrals and Chen's de Rham Theorem for the fundamental group.
4. Hodge theory for the fundamental groupoid.

Future directions:

- Multiple zeta values.
- Polylogrithms.
- Motivic Cohomology.
- Mixed Motives.