Arthur-Selberg Trace Formula: An introduction

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Outline

Introduction

Although the title of this seminar is "local trace formula", i.e., a local analogue of the Arthur-Selberg TF that describes the character of the representation of $G(F)$ on the discrete part of $L^2(G(F))$, for G a reductive algebraic group over a local field F, we need to spell out the Arthur-Selberg TF for global fields (or the rings of Adèles for a global field), to appreciate the major innovations in the establishment of the local trace formula.

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Plan for this lecture

We begin by carefully proving the Selberg TF (1956), an expression for the character of the unitary representation of G on the space $L^2(\Gamma \setminus G)$ of square-integrable functions, where G is a Lie group and Γ a cofinite discrete group, i.e. the character is given by the trace of certain functions on G, followed a few examples, then say a bit about its generalization, Arthur-Selberg TF.

Selberg TF for Γ cocompact (1956)

Notations

- $\mathcal{H}$ denotes a locally compact group, Γ a discrete cocompact subgroup, i.e., $\Gamma \setminus \mathcal{H}$ is compact.

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- Since $\mathcal{H}$ is locally compact, with Γ cocompact, there is a Haar measure which is both left and right invariant, i.e. $\mathcal{H}$ is unimodular

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Right regular representation operator R

For $f \in \mathscr{C}_c^\infty(\mathcal{H})$, $\varphi \in L^2(\Gamma \setminus \mathcal{H})$, define a right regular representation operator R:

$$(R(f)\varphi)(x) = \int_{\mathcal{H}} \varphi(xy) f(y)\, dy.$$

The Kernel $K_f(x, y)$

Obtain $K_f(x, y)$

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$$\begin{aligned}(R(f)\varphi)(x) &= \int_{\mathcal{H}} \varphi(xy) f(y)\, dy \\ &= \int_{\mathcal{H}} \varphi(y) f(x^{-1}y)\, dy \\ &= \int_{\Gamma\backslash\mathcal{H}} \left(\sum_{\gamma\in\Gamma} f(x^{-1}\gamma y) \right) \varphi(y)\, dy\end{aligned}$$

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$$\begin{aligned}
(R(f)\varphi)(x) &= \int_{\mathcal{H}} \varphi(xy) f(y)\, dy \\
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&= \int_{\Gamma\backslash\mathcal{H}} \left(\sum_{\gamma\in\Gamma} f(x^{-1}\gamma y) \right) \varphi(y)\, dy \\
:&= \int_{\Gamma\backslash\mathcal{H}} K_f(x,y)\varphi(y)\, dy
\end{aligned}$$

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Since f is compactly supported, the sum is locally finite, and

$$\int_{\Gamma\backslash\mathcal{H}} \int_{\Gamma\backslash\mathcal{H}} |K_f(x,y)|\, dx\, dx < \infty$$

Facts from operator theory

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Recall from operator theory, the following relations among Hilbert spaces:

$$\left\{ \begin{array}{c} \text{bounded} \\ \text{operators} \end{array} \right\} \supset \left\{ \begin{array}{c} \text{compact} \\ \text{operators} \end{array} \right\} \supset \left\{ \begin{array}{c} \text{Hilbert-Schmidt} \\ \text{operators} \end{array} \right\} \supset \left\{ \begin{array}{c} \text{trace class} \\ \text{operators} \end{array} \right\}$$

For every $f \in \mathscr{C}_c^\infty(\mathcal{H})$, $R(f)$ is Hilbert-Schmidt, thus a compact operator.

Facts from operator theory

$$K_f \in L^2(\Gamma \setminus \mathcal{H} \times \Gamma \setminus \mathcal{H}), \text{ a Hilbert-Schmidt space.}$$

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For every $f \in \mathscr{C}_c^\infty(\mathcal{H})$, $R(f)$ is Hilbert-Schmidt, thus a compact operator.

Let $(R, \mathscr{H})$ be a pair consisting of a unitary representation of $\mathcal{H}$ and a Hilbert space, such that $R(f)$ is compact for all $f \in \mathscr{C}_c^\infty(\mathcal{H})$, then

$$\mathscr{H} = \widehat{\bigoplus} \mathscr{H}_\pi, \quad \text{where} \quad \mathscr{H}_\pi = V_\pi \otimes M_\pi,$$

where (π, V_π) is an irreducible unitary representation and M_π an vector space with $\mathcal{H}$-action, and $\dim M_\pi = m_\pi < \infty$. Remark: since R compact, the decomposition is discrete with finite multiplicity m_π.

Spectral side of the Selberg TF (1/2)

Corollary to the above

$$L^2(\Gamma \setminus \mathcal{H}) = \widehat{\bigoplus}(V_\pi \otimes M_\pi)$$

discrete sum of irreducible representations, thus it only has a discrete spectrum.

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Any $f \in \mathscr{C}_c^\infty(\mathcal{H})$ if of the form

$$f = \sum_{i \text{ finite}} h_i * g_i, \quad \text{for } h_i, g_i \in \mathscr{C}_c^\infty(\mathcal{H}).$$

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Now we can factorize R as

$$R(f) = \sum_{i \text{ finite}} R(h_i)R(g_i),$$

since both $R(h_i)$ and $R(g_i)$ are Hilbert-Schmidt, their product is trace class and hence $R(f)$, a finite sum of trace class operators is trace class.

Spectral side of the Selberg TF (2/2)

Spectral side of Selberg TF

$$\mathrm{tr} R(f) = \sum_{\pi \in \widehat{\mathcal{H}}} m_\Gamma(\pi) \mathrm{tr} \pi(f),$$

where $\widehat{\mathcal{H}}$ the dual of $\mathcal{H}$, i.e., the set of equivalence classes of irreducible representations of $\mathcal{H}$ in this case.

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a word about $\pi(f)$

for a representation (π, V_π) of $\mathcal{H}$, V_π is a Hilbert space, for $v \in V_\pi$ and $x \in \mathcal{H}$,

$$\pi(f) := \int f(x)\pi(x)\,dx.$$

Geometric side of the Selberg TF (1/2)

Exercise

Assume that a kernel $K(x, y)$ on $X \times X$ is

1. continuous in x, y,
2. represents a trace class operator,

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If you are too lazy to do the above...

If

$$K(x,y) = \int_X K_1(x,z)\overline{K_2(z,y)}\, dz$$

with K_i, $i = 1, 2$ Hilbert-Schimdt. Then

$$\mathrm{tr} K = \int_X K(x,x)\, dx = \int_X \int_X K_1(x,z)\overline{K_2(z,x)}\, dz\, dx = \langle K_1, K_2^* \rangle_{L^2(K_1,K_2)}.$$

Geometric side of the Selberg TF (2/2)

$$\begin{aligned}
\mathrm{tr}(R(f)) &= \int_{\Gamma\backslash\mathcal{H}} K(x,x)\,dx \\
&= \int_{\Gamma\backslash\mathcal{H}} \sum_{\gamma\in\Gamma} f(x^{-1}\gamma x)\,dx \\
&= \int_{\mathcal{F}} \sum_{\gamma\in\{\Gamma\}} \sum_{\xi\in\Gamma_\gamma\backslash\Gamma} f(x^{-1}\xi^{-1}\gamma\xi x)\,dx \\
&= \sum_{\gamma\in\{\Gamma\}} \int_{\mathcal{F}} \sum_{\xi\in\Gamma_\gamma\backslash\Gamma} f(x^{-1}\xi^{-1}\gamma\xi x)\,dx \\
&= \sum_{\gamma\in\{\Gamma\}} \int_{\Gamma_\gamma\backslash\mathcal{H}} f(x^{-1}\gamma x)\,dx \\
&= \sum_{\gamma\in\{\Gamma\}} \mathrm{vol}(\Gamma_\gamma\setminus\mathcal{H}_\gamma) \int_{\mathcal{H}_\gamma\backslash\mathcal{H}} f(x^{-1}\gamma x)\,dx
\end{aligned}$$

The Selberg TF and (familiar) examples

Selberg TF for Γ cocompact

For $f \in \mathscr{C}_c^\infty(\mathcal{H})$,

$$\sum m_\Gamma(\pi)\mathrm{tr}\pi(f) = \sum_{\gamma\in\{\Gamma\}} \mathrm{vol}(\Gamma_\gamma \setminus \mathcal{H}_\gamma) \int_{\mathcal{H}_\gamma\setminus\mathcal{H}} f(x^{-1}\gamma x)\, dx$$

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(Familiar) examples

1. $\Gamma = \{1\}, \mathcal{H}$ compact, then the Selberg TF reads

$$\sum m(\pi)\mathrm{tr}\pi(f) = f(1),$$

with $m(\pi) = \dim \pi$, i.e., it is just the usual Plancherel formula.

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2. Γ cocompact, $\mathcal{H}$ abelian, then the Selberg TF is just the Poisson summation formula (spell it out yourself!).

When $\mathcal{H} = \mathrm{SL}(2, \mathbb{R})$

The best known discrete subgroup of $\mathcal{H} = \mathrm{SL}(2, \mathbb{R})$ is the group $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ of unimodular integral matrices. However, the quotient $\Gamma \setminus \mathcal{H}$ is noncompact; nevertheless, Selberg was able to extend his TF to this case by considering the Riemann surface

$$X = \Gamma \setminus \mathcal{H}/K,$$

which becomes a double cosets, where $K = \mathrm{SO}(2, \mathbb{R})$ is a compact orthogonal group, the stabilizer of i under the transitive action of $\mathrm{SL}(2, \mathbb{R})$ the upper half plane by linear fractional transformations. Selberg then analytically continued Eisenstein Series $E(z, s)$ in order to handle the continuous spectrum of $L^2(\Gamma \setminus \mathcal{H})$, and derived a formula for the trace of $R(f)$ in the space of cusp forms

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Truncation and modified kernels

The main tools used to handle general noncompact quotient are truncation and modified kernel, due to Arthur.

Adèles and Strong Approximation

Adèles

Recall for a global field F, the ring of adéles $\mathbb{A}_F$, is the restricted direct product of the completions F_v w.r.t. the rings of integers $\mathcal{O}_v$:

$$\mathbb{A}_F := \left\{ (x_v) \in \prod_v F_v : x_v \in \mathcal{O}_v, \text{ for all but finitely places } v \right\}.$$

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Example: When $F = \mathbb{Q}$,

$$\mathbb{A}_{\mathbb{Q}} := \{(x_\infty, x_2, x_3, \ldots\} : x_v \in \mathbb{Q}_v (\forall v \leq \infty), x_p \in \mathbb{Z}_p, \\ \text{(for all but finitely many } p)\}.$$

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Fundamental domain

Let a group G act on a set X (on the left). A fundamental domain for this action is a subset $D \subseteq X$ satisfying:

Adèles and Strong Approximation

1. For each $x \in X$, there exists $d \in D$ and $g \in G$ such that $gx = d$.
2. The choice of d above is unique

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Strong approximation for adeles

Let v_0 be any normalized nontrivial valuation of the global field F. Let $\mathbb{A}_{F,v_0}$ be the restricted topological product of the F_v with respect to the $\mathcal{O}_v$, where v runs through all normalized valuations $v \neq v_0$. Then F is dense in $\mathbb{A}_{F,v_0}$.

Proof: see [Cassels-Frohlich]

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Strong approximation for algebraic groups over F

The strong approximation theorem is an extension of the Chinese Remainder Theorem to algebraic groups over global fields F:
Let G be a linear algebraic group over a global field F. If S is a non-empty finite set of places of F, then we write A^S for the ring of S-adeles, thus $\mathbb{A} = \mathbb{A}^S \times \mathbb{A}_S$. For any choice of S, $G(F)$ embeds in $G(\mathbb{A}^S)$ and $G(\mathbb{A}_S)$.

A Simple Example

With strong approximation, we are thus able to consider $G(F) \setminus G(\mathbb{A}_F)$, where $G(F)$ is discrete in $G(\mathbb{A}_F)$, thus we can try use the Selberg TF, but the quotient is noncompact in general :(.

Baby example of $G(F) \setminus G(\mathbb{A}_F)$ being noncompact

Take $G = \mathrm{GL}(1)$, $F = \mathbb{Q}$, then

$$G(F) \setminus G(\mathbb{A}_F) = \mathbb{Q}^\times \mathbb{A}_\mathbb{Q}^\times = \mathbb{R}_{>0}^\times \qquad \text{noncompact}$$

When G is the multiplicative group of a quaternion algebra D over $\mathbb{Q}$

Take $G = \mathrm{GL}(1)$, then $G(\mathbb{Q}) = \mathbb{Q}^\times, G(\mathbb{A}_\mathbb{Q}) = \mathbb{A}^\times$, the group of ideles. The restriction of the norm mapping N to G is a generator of the group $X(G)_\mathbb{Q}$, and

$$G(\mathbb{A})^1 := \{x \in G(\mathbb{A}) : |N(x)| = 1\}, \text{ the norm } 1 \text{ elements in } D^\times.$$

Now the quotient $G(\mathbb{Q}) \setminus G(\mathbb{A})^1$ is compact, since G has no proper parabolic subgroup over $\mathbb{Q}$, thus we are able to use the Selberg TF.

Example: Jacquet-Langlands correspondence

Langlands developed a theory of Eisenstein series valid for any reductive group G, and used it to describe the continuous spectrum of $L^2(G(F) \setminus G(\mathbb{A}))$, and formed the Langlands program, as a typical example in this program:

Let $\mathcal{H} = G'(\mathbb{A}_F)/A_{G'}$, D be a division quaternion algebra over a number field F and $G' = D^\times$. Then to each automorphic cuspidal representation σ of G' (i.e., an irreducible $G'(\mathbb{A})$-submodule of $L^2(Z(A)G'(F) \setminus G'(\mathbb{A}))$ of dimension greater than one) there exists a corresponding automorphic cuspidal representation $\pi = \pi(\sigma)$ of $G(\mathbb{A}) = \mathrm{GL}_2(\mathbb{A})$ (an irreducible $G(\mathbb{A})$-submodule of the space of cusp forms $L_0^2(Z(\mathbb{A})G(F) \setminus G(\mathbb{A}))$ with the property that for each place v in F unramified for D (i.e. where $G'(F_v) \cong \mathrm{GL}_2(F_v)), \sigma_v \cong \pi_v$. $Z(D^\times) \cong \mathbb{Z}(\mathrm{GL}_2)$

If you are not comfortable with adeles over global fields, take $F = \mathbb{Q}$, then the places v are either ∞ or a finite prime p.

Noncompact quotient and parabolic subgroups (1/3)

Difficulties with noncompact quotient

When $\Gamma \setminus G$ is non-compact,

1. Although $R(f)$ is still an integral operator, with kernel

$$K_f(x,y) = \sum_{\gamma} f(x^{-1}\gamma y)$$

   no longer integrable over the diagonal.

2. The regular representation $R(g)$ no longer decomposes discretely, Eisenstein series are required to describe the continuous spectrum, and $R(f)$ is of course no longer of trace class.

Arthur's contribution

Arthur derives a form of TF, that still relates geometric and spectral distributions attached to G, by truncating and modifying the kernel.

Noncompact quotient and parabolic subgroups (2/3)

For

$$J_{\mathfrak{o}}^T(f) = \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} K_{\mathfrak{o}}^T(x,x)\,dx,$$

with $K_{\mathfrak{o}}^T$ a modification of the kernel,

$$K_{\mathfrak{o}}(x,x) = \sum_{\gamma\in\mathfrak{o}} f(x^{-1}\gamma x),$$

which can be integrated over the diagonal.

Example with $G = \mathrm{GL}(2)$ (for necessity of truncation and modified kernel)

We have $\mathfrak{o}$ the hyperbolic conjugacy class

$$\mathfrak{o} = \left\{\delta^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} : \delta \in M(F)\setminus G(F)\right\} \text{ with } M = \left\{\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix}\right\}.$$

Noncompact quotient and parabolic subgroups (3/3)

$$
\begin{aligned}
J_{\mathfrak{o}}(f) &:= \int_{Z(\mathbb{A}_F)G(F)\backslash G(\mathbb{A})} K_{\mathfrak{o}}(x,x)\,dx \\
&= \cdots \\
&= m(\mathbb{Z}(\mathbb{A}_F)M(F) \setminus M(\mathbb{A}_F))F_f\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right),
\end{aligned}
$$

but it is infinite as $m(Z(\mathbb{A}_F)M(F) \setminus M(\mathbb{A}_F)) = m(F^\times \setminus \mathbb{A}_F^\times) = \infty$.

A word on the spectral side

$$
L^2(Z(\mathbb{A}_F)G(F) \setminus G(\mathbb{A}_F) = \sum_{\chi \in \mathfrak{X}} L^2_\chi,
$$

with each L^2_χ a $G(\mathbb{A}_F)$-invariant submodule indexed by certain cuspidal data $\mathfrak{X} = \{(M, \sigma)\}$, where M is a Levi subgroup of a parabolic $P \subset G$, and σ is a cuspidal automorphic representation of $M(\mathbb{A}_F)$.

Arthur's modified kernels: The geometric terms (1/2)

For $f \in \mathscr{C}_c^\infty(Z(\mathbb{A}_F) \setminus G(\mathbb{A}_F))$, set

$$K_{\mathfrak{o}}(x,y) = \sum_{\gamma \in \mathfrak{o}} f(x^{-1}\gamma y)$$

and let $\mathfrak{O}$ denote the collection of all such classes $\mathfrak{o}$ in $Z(\mathbb{A}) \setminus G(\mathbb{A})$.
Note: for $\mathfrak{o}$ an elliptic class, (i.e. if $\gamma \in \mathfrak{o}$ is not conjugate in $G(F)$ to an element of any proper parabolic subgroup $P(F)$), the kernel $K_{\mathfrak{o}}(x,y)$ is integrable over the diagonal subset $Z(\mathbb{A}_F)G(F) \setminus G(\mathbb{A}_F)$.

For $P = B$, the Borel subgroup, let $\widehat{\tau}_B$ denote the characteristic function of the positive Weyl chamber

$$\mathfrak{a}_B^+ := \{(r_1, r_2) \in \mathfrak{a}_B = \mathfrak{a}_N\}$$

For any $T = (T_1, T_2) \in \mathfrak{a}_B^+$,

$$k_{\mathfrak{o}}^T(x,f) = K_{\mathfrak{o}}(x,x) - \sum_{\delta(B(F)\setminus G(F)} K_{B,\mathfrak{o}}(\delta x, \delta x)\widehat{\tau}_B(H(\delta x) - T),$$

Arthur's modified kernels: The geometric terms (2/2)

where

$$K_{B,\mathfrak{o}}(x,y) = \sum_{\delta \in \mathfrak{o} \cap Z(F) \backslash M(F)} \int_{N(\mathbb{A}_F)} f(x^{-1}\gamma n y)\, dn.$$

Observe that

$$k_{\mathfrak{o}}^T(x,f) = K_{\mathfrak{o}}(x,x),$$

for x in a compact (modulo $Z(\mathbb{A})$) set Ω (how large depends on T). I. e., for x in an appropriate such set Ω, $\widehat{\tau}_B(H(\delta x) - T)$ is identically zero.

Integrability condition for $k_{\mathfrak{o}}^T(k,f)$, with $G = \mathrm{GL}(2)$ in mind

-For any $\mathfrak{o} \in \mathfrak{O}, k_{\mathfrak{o}}^T(x,f)$ is absolutely integrable over $Z(\mathbb{A})G(F) \setminus G(\mathbb{A})$.
-For $\alpha(T)$ sufficiently large,

$$\sum_{\mathfrak{o} \in \mathfrak{O}} \int_{Z(\mathbb{A})G(F) \backslash G(\mathbb{A})} |k_{\mathfrak{o}}^T(x,f)|\, dx < \infty.$$

Arthur's modified kernels: The spectral terms (1/2)

Spectral expansion of the kernel

Recall the spectral expansion of the kernel

$$K_f(x,y) = \sum_{\chi \in \mathfrak{X}} K_\chi(x,y).$$

Truncation operator on $G(F) \setminus G(\mathbb{A}_F)$

Given $T \in \mathfrak{a}^+$ as before, the truncation of a continuous function $\varphi(x)$ on $Z(\mathbb{A})G(F) \setminus G(A)$ is the function

$$\Lambda^T \varphi(x) = \varphi(x) - \sum_{\delta \in B(F) \setminus G(F)} \varphi_N(\delta x) \widehat{\tau}_B(H(\delta x) - T)),$$

where $\varphi_N(\delta x) = \int_{N(F)\setminus N(\mathbb{A})} \varphi(nx)\, dx$ the constant term of φ.

Arthur's modified kernels: The spectral terms (2/2)

Modified spectral kernel

With identity

$$\sum_{\mathfrak{o}\in\mathfrak{O}} K_{P,\mathfrak{o}}(x,y) = \sum_{\chi\in\mathfrak{X}} K_{P,\chi}(x,y),$$

Arthur defines the modified spectral kernel functions

$$k_{\mathfrak{o}}^T(x,f) = \sum_{P\subset G} (-1)^{\dim A_p/A_G} \sum_{\delta\in P(F)\backslash G(F)} K_{P,\chi}(\delta x,\delta x)\widehat{\tau}_P(H_P(\delta x)-T),$$

For $G = \mathrm{GL}(2)$

$$k_{\chi}^T(x,f) = K_{\chi}(x,x) - \sum_{\delta\in B(F)\backslash G(F)} K_{B,\chi}(\delta x,\delta x)\widehat{\tau}_B(H_B(\delta(\delta x)-T).$$

Because of the above identity, the modification of the geometric expression for the kernel of $R(f)$ is equal to the modification of the spectral expression for this kernel.

Arthur-Selberg Trace Formula

Arthur Trace Formula

With

$$J_{\mathfrak{o}}^T = \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} k_{\mathfrak{o}}^T(x, f)\, dx \quad \text{and} \quad J_\chi^T = \int k_\chi^T(x, f)\, dx,$$

Arthur proved

$$\sum_{\mathfrak{o} \in \mathfrak{O}} J_{\mathfrak{o}}^T(f) = \sum_{\chi \in \mathfrak{X}} J_\chi^T(f).$$

Trace formula explicitly

$$\sum_M \frac{1}{|W(M)|} \int_{\pi(M,V)} a^M(\pi) I_M(\pi, f)\, d\pi = \sum_M \frac{1}{|W(M)|} \sum_{\gamma \in \Gamma(M,V)} a^M(\gamma) I_M(\gamma, f)$$

Recap and Future topics

Review

Today we discussed

1. a proof of Selberg TF.
2. (and reviewed) the ring of adeles and strong approximation for algebraic groups over a global field F.
3. a few nice applications of the TF.
4. the difficulties for obtaining a TF over noncompact quotients.
5. the statement of Arthur TF, with $G = \mathrm{GL}(2)$ as an example in mind.

Possible future topics

1. a careful dissuasion on roots and weights.
2. Eisenstein series for the necessary proofs.
3. orbital integrals and a detailed proof of Arthur TF.
4. Shalika germs (in local TF).