Selberg Trace Formula for Compact Quotients

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Outline

Introduction

The title of this seminar indicates that we hope to reach the goal of understanding Arthur's Trace Formula, a beefed up version of the Poisson summation formula for non-compact quotient groups. Before we try to understand Arthur's Trace Formula, we examine its predecessor, the Selberg Trace Formula (1956) for compact quotients.

Plan for this lecture

1. Define Haar measures on locally compact topological groups and unimodular groups.
2. Define right regular representations and recall integral kernels.
3. Examine the spectral side of the Trace Formula.
4. Examine the geometric side of the Trace Formula.
5. State the Selberg Trace Formula.

Locally Compact Topological Groups

Topological groups

A **topological group** G is a group that is also a topological space, having the property that multiplication $(g_1, g_2) \mapsto g_1 g_2$, and inversion $g \mapsto g^{-1}$ are both continuous.
A topological group G is a **locally compact group** if G is locally compact as a topological space.

Haar measure

A **left invariant Haar measure** on G is a measure $\mu : B \to [0, \infty)$, where B is a σ algebra containing all Borel sets of G, such that

- $\mu(K) < \infty$ for any compact set $K \in B$,
- $\mu(gS) = \mu(S)$ for all $g \in G$ and $S \in B$,
- Every Borel set E is outer regular,
- Every open set E is inner regular.

Selberg TF for Γ cocompact (1956)

Notations

- G denotes a locally compact group, Γ a discrete cocompact subgroup, i.e., $\Gamma \setminus G$ is compact.
- $L^2(\Gamma \setminus G)$ is thus well defined with a action G.
- $\mathscr{C}_c^c(G)$ denotes the set of continuous, compactly supported functions on G.
- Since G is compact, with Γ cocompact, there is a Haar measure which is both left and right invariant, i.e. G is **unimodular**.

Right regular representation operator R

For $f \in \mathscr{C}_c^c(G)$, $\varphi \in L^2(\Gamma \setminus G)$, define a right regular representation operator R:

$$(R(f)\varphi)(x) = \int_G \varphi(xy) f(y)\, dy.$$

The Kernel $K_f(x,y)$

Obtain $K_f(x,y)$

$$
\begin{aligned}
(R(f)\varphi)(x) &= \int_G \varphi(xy) f(y)\, dy \\
&= \int_G \varphi(y) f(x^{-1}y)\, dy \\
&= \int_{\Gamma\backslash G} \left(\sum_{\gamma\in\Gamma} f(x^{-1}\gamma y) \right) \varphi(y)\, dy \\
:&= \int_{\Gamma\backslash G} K_f(x,y)\varphi(y)\, dy
\end{aligned}
$$

Since f is compactly supported, $K_f(x,y)$ converges, and

$$
\int_{\Gamma\backslash G} \int_{\Gamma\backslash G} |K_f(x,y)|\, dx\, dx < c
$$

Recall facts from operator theory

$$K_f \in L^2(\Gamma \setminus G \times \Gamma \setminus G), \text{ a Hilbert-Schmidt space.}$$

Recall from operator theory, the following relations among Hilbert spaces:

$$\left\{ \begin{array}{c} \text{bounded} \\ \text{operators} \end{array} \right\} \supset \left\{ \begin{array}{c} \text{compact} \\ \text{operators} \end{array} \right\} \supset \left\{ \begin{array}{c} \text{Hilbert-Schmidt} \\ \text{operators} \end{array} \right\} \supset \left\{ \begin{array}{c} \text{trace class} \\ \text{operators} \end{array} \right\}$$

For every $f \in \mathscr{C}_c^c(G)$, $R(f)$ is Hilbert-Schmidt, thus a compact operator

Let $(R, \mathscr{H})$ be a pair consisting of a unitary representation of G and a Hilbert space, such that $R(f)$ is compact for all $f \in \mathscr{C}_c^c(H)$, then

$$\mathscr{H} = \widehat{\bigoplus} \mathscr{H}_\pi, \quad \text{where} \quad \mathscr{H}_\pi = V_\pi \otimes M_\pi,$$

where (π, V_π) is an irreducible unitary representation and M_π an vector space with action, and $\dim M_\pi = m_\pi < c$. Remark: since R compact, the decomposition is discrete with finite multiplicity.

Spectral side of the Selberg TF (1/2)

Corollary to the above

$$L^2(\Gamma \setminus G) = \widehat{\bigoplus}(V_\pi \otimes M_\pi)$$

discrete sum of irreducible representations, thus it only has a discrete spectrum.

Any $f \in \mathscr{C}_c^c(G)$ if of the form

$$f = \sum_{i \text{ finite}} f_i * f_i', \quad \text{where } f_i, f_i' \in \mathscr{C}_c^c(G).$$

Now we can factorize R as

$$R(f) = \sum_{i \text{ finite}} R(f_i) R(f_i'),$$

since both $R(f_i)$ and $R(f_i')$ are Hilbert-Schmidt, their product is trace class and hence $R(f)$, a finite sum of trace class operators is trace class.

Spectral side of the Selberg TF (2/2)

Spectral side of Selberg TF

$$\mathrm{tr}R(f) = \sum_{\pi \in \widehat{G}} m_\Gamma(\pi)\mathrm{tr}\pi(f)$$

where $\widehat{G}$ the dual of G, is the set of equivalence classes of irreducible representations of G in this case.

a word about $\pi(f)$

for a representation (π, V_π) of G, V_π is a Hilbert space, for $v \in V_\pi$ and $x \in G$,

$$\pi(f) := \int f(x)\pi(x)\, dx.$$

Geometric side of the Selberg TF (1/2)

A Theorem from operator theory

Assume that a kernel $K(x, y)$ on $X \times X$ is

1. continuous in x, y,
2. represents a trace class operator,

then

$$\mathrm{tr} R = \int_X K(x, x)\, dx.$$

If you are too lazy to do the above...

If

$$K(x, y) = \int_X K_1(x, z)\overline{K_2(z, y)}\, dz$$

with K_i, $i = 1, 2$ Hilbert-Schmidt. Then

$$\mathrm{tr} K = \int_X K(x, x)\, dx = \int_X \int_X K_1(x, z)\overline{K_2(z, x)}\, dz\, dx = \langle K_1, K_2 \rangle_{L^2(K_1, K_2)}.$$

Geometric side of the Selberg TF (2/2)

$$
\begin{aligned}
\operatorname{tr}(R(f)) &= \int_{\Gamma\backslash G} K(x,x)\,dx \\
&= \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(x^{-1}\gamma x)\,dx \\
&= \int_{\Gamma\backslash G} \sum_{\gamma\in\{\Gamma\}} \sum_{\xi\in\Gamma_\gamma\backslash\Gamma} f(x^{-1}\xi^{-1}\gamma\xi x)\,dx \\
&= \sum_{\gamma\in\{\Gamma\}} \int_{\Gamma\backslash G} \sum_{\xi\in\Gamma_\gamma\backslash\Gamma} f(x^{-1}\xi^{-1}\gamma\xi x)\,dx \\
&= \sum_{\gamma\in\{\Gamma\}} \int_{\Gamma_\gamma\backslash G} f(x^{-1}\gamma x)\,dx \\
&= \sum_{\gamma\in\{\Gamma\}} \operatorname{vol}(\Gamma_\gamma \setminus G_\gamma) \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)\,dx
\end{aligned}
$$

The Selberg TF and (familiar) examples

Selberg TF for Γ cocompact

For $f \in \mathscr{C}_c^c(G)$,

$$\sum m_\Gamma(\pi)\mathrm{tr}\pi(f) = \sum_{\gamma \in \{\Gamma\}} \mathrm{vol}(\Gamma_\gamma \setminus G_\gamma) \int_{G_\gamma \setminus G} f(x^{-1}\gamma x)\, dx$$

(Familiar) examples

1. $\Gamma = \{1\}, G$ compact, then the Selberg TF reads

$$\sum m(\pi)\mathrm{tr}\pi(f) = f(1),$$

   with $m(\pi) = \dim \pi$, i.e., it is just the usual Plancherel formula.

2. Γ cocompact, G abelian, then the Selberg TF is just the Poisson summation formula.

Problems with G being non-compact and Future topics

Problems with non-compact G and the remedy

1. The kernel
$$K_f(x,y) = \sum_{\gamma} f(x^{-1}\gamma y)$$
is **no longer integrable** over the diagonal.
The remedy: **modify this kernel**.
2. The regular representation $R(f)$ **no longer decomposes discretely**, **Eisenstein series** are required to describe the **continuous spectrum**, and $R(f)$ is of course no longer of trace class.
The remedy: **truncation**.

Future topics

1. Eisenstein series and intertwining operators.
2. Non-compact quotient and parabolic subgroups.
3. Modified kernels and Arthur TF.